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Nonlocal Structures: Bilocal Photon

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As a starting point, it is postulated that all particles and fields are built from a single primitive field, which must then be a massless fermion with a σ spin of one-half. Two helicities are embodied in a τ spin of one-half. The vacuum is an open Fermi sea whose height is a wave number κ. Elementary particles are structures having the form of standing-wave systems floating on the vacuum sea, with the height κ providing both the scale of inner structural size and the mass unit for the elementary particle mass spectrum. A bilocal photon starts with a function describing two primitive quanta with parallel σ spin and opposite τ spin. A centroid-time wave equation then couples-in an infinite set of orthogonal functions. The introduction of an operator Q_λ permits the reduction of the infinite secular determinant to a finite six-by-six determinant. Solutions (for the infinite expansion) are obtained describing photons with right-handed and left-handed polarizations. Superpositions of these give linearly polarized photons. Electric and magnetic field vectors, satisfying the vacuum Maxwell equations, are obtained from a bilocal Hertz vector given by $\mathbf{\Pi} = (2/\kappa^3 c)(\partial/\partial t_r)\nabla_r\Psi(1,2)$, where $\Psi(1,2)$ is the bilocal wave function, and t_r and $\mathbf{r}$ are the relative time and relative position variables.

1. INTRODUCTION

The electron and the muon differ in mass, but otherwise appear to be very much alike. The mass difference suggests that there is a difference in inner structural details, while the great similarity with respect to other properties suggests that the two structures are basically similar in character.

When compared to other particles such as the neutron and proton, the electron and muon are seen to be very much simpler and indeed to be almost the simplest particles that we know of. But their existence, side by side, and their great similarity, make it difficult to call one of them

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fundamental and the other one derived. It seems more logical to expect that each is a structure built from a more primitive entity or entities.

As to a more primitive entity, we have no obvious candidate among observed particles. However, the quarks which are useful in understanding nucleons and other hadrons have not yet been observed and may never be. The primitive entity needed for understanding leptons may be similarly unobservable in isolation from a structure.

The properties to be associated with a primitive entity are not obvious if we think only of the electron and muon. When we put these two particles into a larger framework, we will be able to make more progress.

Our knowledge of the structure of matter has progressed by successive stages of inner structure. The molecule is composed of atoms. These were found to be built from nuclei surrounded by electrons. The nuclei are groups of neutrons and protons held together by fields in which the pions play an important role. Each nucleon is a distributed structure characterized by experimentally measured form factors.

This historical sequence has been an uncovering of successive levels of inner structure. Should we expect the sequence to go on forever? Or will it terminate, and if so then how can we be sure it has terminated?

If we reach a level with only one constituent, from which all higher levels are derivable, then this level is effectively the last and lowest level. It makes no logical sense to ask what the one constituent is built from, if the one constituent always appears as a unit in the construction of all the elementary particles and fields with which we deal directly in our observations.

Because of the simplicity and similarity of the electron and muon, we can hope that a next level dealing with their inner constituents will not be far from a last level, if such exists. Let us postulate, therefore, that there does exist a last level, with only one constituent, and that this constituent is the primitive entity which we need for the construction of the electron and the muon.

We will want to infer the properties of the primitive entity from the postulate that it is the single constituent of the last level. As such, the primitive entity is required to be all that we need to construct all of the known elementary particles and all of the known interaction fields.

No loss of generality follows if we describe the primitive entity as a primitive field, ξ. From the known existence of photons and neutrinos, the primitive field must be massless. From the known existence of particles which are bosons of integral spin, and other particles which are fermions of half-integral spin, the primitive field must be a fermion field with half-integral spin, most assuredly a spin of one-half.

2. THE VACUUM SEA

Since all known elementary particles have both a wave aspect and a particle aspect, it can be inferred that the primitive field, ξ, must also have both a wave aspect and a particle aspect. Since ξ is a fermion field, the exclusion principle will apply to its quanta. Since ξ is massless, there can be only a limited sort of localization of its quanta in space.

The universe has a substantial density of elementary particles, each postulated to be constructed from ξ quanta. There are accordingly a large number of ξ quanta in the universe, and these obey Fermi–Dirac statistics. Because of the exclusion principle, these massless quanta will have to build up into a Fermi sea in those intermediate regions where they would otherwise attempt to occupy the same low-lying quantum levels.

When we try to build a structure with rest mass out of massless ingredients, we are forced to represent the structure, when at rest, as a standing-wave system in which there are incoming converging waves of the primitive field which enter a central region, participate in exchange interactions, then leave as outgoing diverging waves. The structure itself is a crossroads characterized by a cloverleaf mechanism which can have great complexity and special kinds of symmetry. The massless waves are always passing through, bringing information from the other crossroads centers they have just passed through earlier, and then carrying away information to the other crossroads centers towards which they are heading.

In its inner realm, the structure can be exceedingly complicated, though we would hope to find the inner realm of the electron and muon to be relatively simple. Any structure, however, in its outer realm, must become a fairly simple combination of incoming and outgoing waves of the massless primitive field, ξ.

The vacuum which separates different elementary particles will be the carrier of gravitational interaction between these particles, no matter what kind of particles are involved. Our initial postulate requires that the primitive field convey all interactions, and we must include gravitational interactions. Each structure has incoming and outgoing massless waves, and it would be appropriate that these waves, which must always be present, should carry the gravitational interaction, which similarly must always be present.

In a region where the density of elementary particles is large, the vacuum which separates these particles will be crisscrossed by a large number of primitive waves heading to and from these particles. The exclusion principle will require that this vacuum be characterized by a minimum Fermi height which is the wave number required to accommodate these primitive waves within the simplest available Fermi sea.

In a region where the density of elementary particles is smaller, a particular point in the vacuum separating these particles will be traversed by fewer primitive waves, and a lower Fermi height will be adequate to accommodate the passing primitive waves within the simplest available Fermi sea.

We can be led to a picture of the ordinary vacuum as a Fermi sea of primitive quanta, whose Fermi height is large where the density of elementary particles is great, but smaller where the density of particles is smaller. The Fermi height then becomes a measure of the gravitational potential, and we would expect that a gradient in this Fermi height would correspond to a gravitational force, leading to the acceleration of an elementary particle.

3. BOUNDARY CONDITIONS

We can fit a complicated structure into a Fermi sea if the quanta in the structure have wave numbers that are just above the Fermi surface. The inner realm in the structure can have many forms, as long as the outer parts are representable as incoming and outgoing waves which float on the Fermi surface of the vacuum sea. This is a boundary condition that a structure needs to satisfy if it is to represent an elementary particle that is in equilibrium with the vacuum that surrounds it.

The Fermi height of the vacuum sea will be denoted by the radian wave number κ. When a structure is forced to satisfy the boundary condition described above, with its outer regions merging into the Fermi surface, the wave number κ will set the scale of size for the inner details of the structure. At the same time, the wave number κ will serve as a mass unit, in terms of which the rest mass of the structure will be given as some numerical multiple (probably not an integer multiple). In a part of space where κ is large, all of the masses of the elementary particles will be large, since κ serves as the mass unit for the entire elementary particle mass spectrum.

The wave number κ will be large in a region of the vacuum that is close to a large number of elementary particles. Such a region might be the surface of a heavy star, or a point near the center of a galaxy. In such a region, all elementary particles are more massive than they are here at the earth's surface, according to our picture. This picture is in close accordance with Mach's principle, which attributes the inertial properties of a particle (including rest mass) to its gravitational interaction with the rest of the universe. Greater interaction should mean greater mass.

In a region to one side of a concentration of elementary particles, there will be a spatial gradient of the Fermi height κ. A structure built from primitive quanta which is required to match one value of κ on the left, and another value of κ on the right, will be able to do this if it is undergoing an acceleration given by

$$\mathbf{g} = c^2 \nabla(\log_e \kappa) \tag{3.1}$$

where c is the velocity associated with the primitive waves constituting the structure, and where $\mathbf{g}$ is the acceleration from a resting start.

4. OPEN FERMI SURFACE

The familiar form of the Fermi surface is based on a closed box which confines a certain number of quanta. This box may for example be the rectangular boundary surfaces of a piece of copper, while the fermion quanta are the conduction electrons contained within this metallic block.

The conduction electrons can be identified with the available modes of excitation of a rectangular cavity. The available modes are filled up in order of increasing energy, until the available conduction electrons are all allocated. The wave number reached at this point is then the wave number which specifies the Fermi surface.

When we replace the closed box by an open volume, we will need to generalize the concept of a Fermi surface. The problem will be to allow for the absence of side walls, and the corresponding absence of a preferred frame of reference for the observer who is describing the waves in the Fermi sea.

One option which opens up, when the walls of the box are removed, is the replacement of a plane-wave Fermi sea by a sea of spherical waves, converging toward and diverging from a lattice of points spanning the space. If these focal points are at rest with respect to the observer, then symmetry requires that the converging and diverging waves have the same wave number, namely, the wave number κ specifying the surface of the Fermi sea.

If the observer is now set into motion with respect to the above lattice of points of focus, then in the observer's new coordinate frame the converging and diverging waves will be modified by the relativistic Doppler shift so that the wave number to be associated with a particular wave train will depend on the direction of motion of that wave train with respect to the observer's velocity. No longer will a single wave number suffice to characterize the whole of a converging or diverging wave structure. To this moving observer, the Fermi surface is displaced in vector wave number space and is now direction dependent. The Fermi height is now a correlation between wave number and wave direction.

We want to be able to specify an open Fermi surface in a general way that is not changed by the introduction of a moving observer. To do this, we first note that in the fixed-observer picture, with waves in the sea converging to a point and then diverging from this same point, there will be a definite relationship between the phase and amplitude of the converging wave and the phase and amplitude of its diverging companion. This relationship is required to avoid a singularity at the focal point. In effect, the converging and diverging parts belong to a single wave, which is not fully characterized unless both parts are specified.

If we let a single wave function specify both parts of this single wave, then we will be able to deduce the parameters of any Lorentz transformation which changed these two parts from oppositely directed waves with the same wave number, into modified waves which may differ both in wave number and in propagation direction. In this way we can reconstruct the single wave number which characterized the isotropic Fermi sea, and determine whether the wave function does in fact describe a pair of waves to be associated with a Fermi sea whose height is the wave number κ.

Actually, the same reconstruction can be carried out using the plane-wave picture of the Fermi sea. When the observer is at rest with respect to the sea, or to an oppositely directed pair of waves in the sea, then a wave function describing such a pair of waves will be particularly simple in form. The introduction of an observer's velocity will change the two waves via a Lorentz transformation, so that the wave function describing the pair of waves becomes more complicated. However, if an inverse transformation can bring the pair of waves back to the form of an oppositely directed pair with the same wave number κ, then this wave function can be identified as a wave function describing a linked pair of waves on the open Fermi surface.

Conversely, an open Fermi surface requires that each wave function for that surface describe a linked pair of waves, reducing in the simplest case to an oppositely directed pair of waves with the same wave number κ, where κ is the height of the open Fermi surface.

This generalization of the concept of Fermi height is an exploratory one. As we proceed, the description of the vacuum will become more quantitative, and the results obtained should help to clarify this concept.

5. SPINOR FORMALISM

The primitive field must be a spinor field, with a σ spin of one-half. But the primitive field is massless, and this leaves it alternative helicities, right-handed and left-handed.

These helicities will be represented by two choices for τ spin. Right-handed helicity will be represented by $\tau_\xi = +1$, while left-handed helicity is represented by $\tau_\xi = -1$. The product operator, $\tau_\xi \boldsymbol{\sigma}$, then becomes the operator representing the forward momentum.

A particular choice of σ spin, then, gives two alternative choices for helicity and hence momentum, and these are oppositely directed. In the rest situation, therefore, a pair of oppositely directed primitive waves can be represented by a wave function which includes two components with the same σ spin and with each choice of τ spin. In the representation with σ_z diagonal, and with τ_ξ diagonal, there will in general be four components for a particular wave function, but only two components will be nonzero if the σ spin is parallel to the z axis.

In the more general case, with the σ spin in an arbitrary direction, all four components of the wave function will be nonzero. A primitive quantum in the vacuum must be represented by a pair of primitive waves, as emphasized above, reducing in the simplest case to a pair of oppositely directed waves of the same wave number κ.

The combined σ-spin and τ-spin formalism can be represented by a four-component framework. This framework can be expressed as

$$\xi = \xi^{\tau\sigma} = \begin{bmatrix} \xi^{++} \\ \xi^{+-} \\ \xi^{-+} \\ \xi^{--} \end{bmatrix}^{\tau\sigma} = \begin{bmatrix} (\kappa/2)^{1/2}\cos(\theta/2)\exp[i(-\varphi+\chi+\eta)/2] \\ (\kappa/2)^{1/2}\sin(\theta/2)\exp[i(\varphi+\chi+\eta)/2] \\ (\kappa/2)^{1/2}\cos(\theta/2)\exp[i(-\varphi-\chi+\eta)/2] \\ (\kappa/2)^{1/2}\sin(\theta/2)\exp[i(\varphi-\chi+\eta)/2] \end{bmatrix}^{\tau\sigma} \tag{5.1}$$

Here the right-hand portion refers to an explicit plane-wave spinor function, but the left-hand parts are more general. The upper two components have positive τ spin, with $\tau_\xi = +1$. The lower two components have $\tau_\xi = -1$. The first and third components have positive σ spin, with $\sigma_z = +1$, while the second and fourth components have $\sigma_z = -1$.

In the right-hand portion of (5.1), the angles θ and φ give the direction of the σ spin. The angle χ is a third Eulerian angle about this σ-spin direction, while $\eta/2$ represents a phase angle common to all four components. We can make this spinor function refer to the surface of the Fermi sea if we specify that

$$\eta/2 = -\kappa ct \tag{5.2a}$$

$$\chi/2 = \boldsymbol{\kappa}\cdot\mathbf{r} \tag{5.2b}$$

where κ is the Fermi height and $\boldsymbol{\kappa}$ is a propagation vector with the magnitude κ and the direction specified by the angles θ and φ. Then the upper two components on the right-hand side of (5.1) describe a wave with the propagation vector $\boldsymbol{\kappa}$, while the lower two components describe a wave with the propagation vector $-\boldsymbol{\kappa}$, propagating in the opposite direction but with the same wave number.

Associated with the spinor framework in (5.1) are the σ spin operators

$$\sigma_x = \boxed{\begin{matrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{matrix}}, \qquad \sigma_y = \boxed{\begin{matrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & i & \cdot \end{matrix}},$$

$$\sigma_z = \boxed{\begin{matrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{matrix}} \tag{5.3}$$

and the τ-spin operators

$$\tau_\xi = \boxed{\begin{matrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{matrix}}, \qquad \tau_\eta = \boxed{\begin{matrix} \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & -i \\ i & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{matrix}},$$

$$\tau_\zeta = \boxed{\begin{matrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{matrix}} \tag{5.4}$$

The expectation value of the vector operator $\boldsymbol{\sigma}$ is given by

$$\bar{\xi}\boldsymbol{\sigma}\xi = \boldsymbol{\kappa} \tag{5.5}$$

where $\bar{\xi}$ is the Hermitian conjugate of ξ. The operator $\tau_\zeta\boldsymbol{\sigma}$ is intended to represent the propagation vector for the system, but there is no single propagation vector here, though there is a single σ-spin vector. We can separate out the propagation vector for the upper two components through the expectation value

$$\bar{\xi}(1+\tau_\zeta)(\tau_\zeta\boldsymbol{\sigma})\xi = +\boldsymbol{\kappa} \tag{5.6a}$$

Similarly, we can separate out the propagation vector for the lower two components through the expectation value

$$\bar{\xi}(1-\tau_\xi)(\tau_\xi\boldsymbol{\sigma})\xi = -\boldsymbol{\kappa} \tag{5.6b}$$

This dichotomy is consistent with the result from (5.2), showing the upper pair of components as representing a wave with the propagation vector $\boldsymbol{\kappa}$, while the lower pair of components represents a wave with the propagation vector $-\boldsymbol{\kappa}$.

Let us now introduce a Lorentz transformation. Let the observer be given a forward velocity $\mathbf{v}$. Associated with this velocity there will be a transformation of the geometrical variables in (5.2a) and (5.2b), but in addition there will be a modification of the wave function (5.1) that is expressible through the substitution

$$\xi_{\mathrm{v}} = \left[(1-\tau_\xi\boldsymbol{\sigma}\cdot\mathbf{v}/c)/(1+\tau_\xi\boldsymbol{\sigma}\cdot\mathbf{v}/c)\right]^{1/4}\xi \tag{5.7}$$

This modification alters the two waves in the wave function (5.1) in such a way that the overtaken wave is Doppler shifted to a smaller wave number, while the other wave in the wave pair is Doppler shifted to a greater wave number. Substitution of (5.7) in the left-hand sides of (5.6a) and (5.6b) leads directly to the appropriately transformed propagation vectors, replacing $+\boldsymbol{\kappa}$ and $-\boldsymbol{\kappa}$ on the right-hand sides.

Since the vacuum sea is intended to be open and unbounded, the Lorentz-transformed wave function (5.7) is as valid as the simpler function (5.1) for the representation of a primitive quantum at the surface of the sea. The transformation in (5.7) is a kind of gauge transformation whose introduction permits us to compensate a wave function for a Lorentz transformation, and in this way to build a Lorentz-invariant Fermi sea for incorporation into a relativistic elementary particle theory.

Given a general plane-wave function of the form ξ_{v}, we can say that this function belongs to the surface of the vacuum sea if it can be transformed by the inverse of (5.7) back to the canonical form (5.1). However, it is not necessary to actually find the transformation. We need only examine two expectation values, which will have the form

$$\bar{\xi}_{\mathrm{v}}\tau_\xi\xi_{\mathrm{v}} = \kappa\cos\beta \tag{5.8a}$$

$$\bar{\xi}_{\mathrm{v}}\tau_\eta\xi_{\mathrm{v}} = \kappa\sin\beta \tag{5.8b}$$

where β is a phase angle which reduces to χ in (5.1) when the ξ function has the canonical form (5.1).

The original σ-spin direction can be reconstructed in a similar way, through

$$\bar{\xi}_v \tau_\xi \boldsymbol{\sigma} \xi_v = \boldsymbol{\kappa} \cos\beta \tag{5.9a}$$

$$\bar{\xi}_v \tau_\eta \boldsymbol{\sigma} \xi_v = \boldsymbol{\kappa} \sin\beta \tag{5.9b}$$

In the canonical form (5.1), the ξ function satisfies a wave equation of the form

$$\left[\frac{1}{i}\tau_\xi \boldsymbol{\sigma}\cdot\nabla + \frac{1}{ic}\frac{\partial}{\partial t}\right]\xi = 0 \tag{5.10}$$

where the explicit dependence on space and time has been included from (5.2). After the transformation (5.7), the wave function ξ_v satisfies the same wave equation, provided that the Lorentz transformation is also applied to the space and time variables in (5.2).

This spinor formalism, using both σ spin and τ spin, will form the basis for the spin dependence of the wave functions we will be building.

6. STRUCTURES

In the discussion of plane waves in the vacuum sea, it was necessary to associate one pair of waves with one sea quantum. We could choose the σ spin to be "up" or "down," but the τ-spin degree of freedom was preempted by the need to allow for the gauge transformation (5.7).

With a spherical-wave picture of the vacuum sea, there is the same requirement that one sea quantum refer to a pair of waves, one converging toward a focal point, the other diverging from this focal point. The exclusion principle will permit two quanta from the vacuum sea to be centered on the same focal point, but will not permit four vacuum quanta, even though we are using a ξ function that has four components.

When we consider a structure such as an electron or muon (or neutron or proton), we can incorporate more quanta per lattice cell, and utilize both τ spin and σ spin for antisymmetrization within the structure. Thus we can have three or four quanta per cell, rather than the limit of two that characterizes the vacuum sea.

The outer portions of the structure must still float on the vacuum sea, but this boundary condition on the structure permits it to have inner complexity if this complexity is sufficiently localized. We can think of the extra quanta within a structure as being interstitials in the vacuum lattice, prevented by the exclusion principle from spreading out into the vacuum away from the structure. The structure needs to have an inner form which confines the interstitials. Antisymmetrization will ensure that all structural quanta share in the outer portions of the structure where the waves merge into the vacuum sea, but these structural quanta have to take turns, since not all of them will be permitted in the sea at the same time. A structure that satisfies these requirements will be an acceptable model for an elementary particle in this theory.

The rest mass of the structure will be proportional to the wave number κ in the surrounding vacuum sea, but the proportionality factor will depend on the details of the structure and will be obtainable from an eigenvalue equation in the mass operator for the structure. If the Fermi height κ is changed, all of the masses in the elementary particle mass spectrum will change in proportion to κ, but mass ratios will be unaffected.

Gravitation has been mentioned earlier, in connection with equation (3.1), where $\log_e \kappa$ plays the role of a gravitational potential. The scaling of structural size is a proper function of a gravitational potential, as is the scaling of the mass spectrum. The Fermi height κ can be considered to act in a way that is similar to the action of the spatial part of the metric, in a metric theory of gravitation.

A metric theory that accounts accurately for the several important gravitational experiments will have a temporal part of the metric that changes in an opposite sense to the spatial part, as the gravitational potential changes. This results in a dependence of the velocity of light upon the gravitational potential, as verified by the measured delay of a radar pulse passing close to the sun's surface (Shapiro et al., 1971).

The velocity of a primitive wave in the vacuum sea should depend in the same way upon the gravitational potential, yet (5.2) leads to a wave velocity c that has been treated as a constant independent of κ.

We can modify this picture to make it consistent with the radar experiment, if we let the wave velocity c in (5.2a) depend upon the Fermi height κ through the relationship

$$c = c_0 \kappa_0^2 / \kappa^2 \tag{6.1}$$

where c_0 and κ_0 are true constants, c_0 being the wave velocity characterizing a reference Fermi sea with the Fermi height κ_0.

We would like to have a wave equation for a single ξ quantum in which we incorporated the interactions with other ξ quanta that would serve to enforce the boundary conditions on a structure representing an elementary particle. As a tentative wave equation, we will generalize from (5.10) by including exchange-interaction terms, leading to the following:

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_i}+\frac{1}{i}\tau_{i\zeta}\boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_i\right]\xi_i+\bar{\xi}_j P_{ij}^{\tau}P_{ij}^{\sigma}\xi_j\xi_i$$

$$+\Lambda\bar{\xi}_j\bar{\xi}_k\frac{1}{2}\left(P_{ijk}^{\tau}P_{jik}^{\sigma}+P_{jik}^{\tau}P_{ijk}^{\sigma}\right)\xi_k\xi_j\xi_i \qquad (6.2)$$

The first part of (6.2) comes directly from (5.10) but has been made to apply to a specific primitive quantum ξ_i. The second part of (6.2) is a pair-exchange term which describes the effect of an encounter between the quantum ξ_i and another primitive quantum ξ_j. The third part of (6.2) is a triple-exchange term in which there is a cyclic exchange of τ spin and σ spin in opposite directions around the trio of quanta, symmetrized by the inclusion of a cyclic exchange with the senses reversed.

The triple-exchange term, describing as it does an encounter between the quantum ξ_i and *two* other quanta, ξ_j and ξ_k, cannot describe a process among just the quanta in the vacuum sea, since no more than two of the sea quanta can be involved in such an encounter. When the ξ wave ξ_i passes through a structure, on the other hand, there can be three-quantum encounters. Thus the triple-exchange term selects only those parts of space containing structures.

For a wave ξ_i passing through a region without structures, the pair-exchange term will keep its parameters in equilibrium with those of the vacuum sea. Where the sea is flat and isotropic, these pair exchanges merely cause role exchanges between the passing quantum and the sea quantum. In this case we can relabel quanta after each pair encounter, and follow a single ξ wave that moves as though the pair-exchange term were not present at all.

Omitting the pair-exchange term from (6.2), and then converting (6.2) to a second-order differential equation by a procedure that removes terms linear in Λ, will give us a second-order wave equation with a source term having the coefficient Λ^2. This source term is nonvanishing only where structures are present.

It is suggestive to consider this second-order wave equation as having the necessary elements to relate the Fermi height in a particular location to the distribution of gravitational sources in neighborhoods near to and far from this location. Equation (5.1) shows how the Fermi height κ is incorporated into the ξ wave, and equation (3.1) gives a tentative relation between the Fermi height and the gravitational potential. This line of argument leads to the conclusion that the squared coefficient Λ^2 should be proportional to the gravitational constant G.

More explicitly, when dimensional considerations are included, we find that

$$\Lambda \sim (Gh/c^3)^{1/2} \doteq 4 \times 10^{-33}\ \text{cm} \tag{6.3}$$

A gravitational source is an energy density, with rest mass included as a form of energy. To get this, we have squared the triple-exchange term in (6.2). We should expect, therefore, to find that the unsquared triple-exchange term in (6.2) represents an assembly of energy amplitudes, quantities which when squared become energy densities. The electric and magnetic field strengths, for example, are energy amplitudes of this character, and ought to be recognizable among the elements contained in a spelled-out version of the triple-exchange term in (6.2).

We will find later that the Fermi height κ has the magnitude

$$\kappa \doteq (2.29701 \times 10^{-13}\ \text{cm})^{-1} \tag{6.4}$$

This wave number is incorporated into the normalization of the ξ quanta, as indicated in (5.1). As a consequence, it can be seen from (6.3) and (6.4) that the triple-exchange term in (6.2) is smaller than the pair-exchange term by a factor of about 10^{20}. This suggests that we associate the "strong interaction" with the pair-exchange term and the "weak interaction" with the triple-exchange term.

We have already suggested that the source for the gravitational potential can be attributed to the square of the triple-exchange term in (6.2). If this proves to be true, then there would seem to be no need for any quadruple-exchange term in the tentative wave equation (6.2). Without including such a term, there is already a framework for including the strong, weak, and gravitational forces. As long as we find the electromagnetic force to be already incorporated in (6.2), we will have covered the presently known interaction forces without requiring any quadruple-exchange term in (6.2).

Further investigation of the interactions between structures can wait until we have explicit structures to work with. The first and simplest of these structures will be the photon.

7. BILOCAL PHOTON

A procedure that we can use for forming structures will be illustrated through the construction of a two-wave structure to represent a photon. To keep the mathematics as simple as possible, we will use a canonical gauge, with the observer at rest in a flat vacuum sea.

Because the photon has no rest mass, we cannot consider using the simplifications available for other structures which can be analyzed while they are at rest with respect to the observer. For the photon, we need to include its own propagation as an inseparable part of its wave function.

We will accordingly include centroid motion in the structure representing the photon, and for simplicity this centroid motion will be taken to be plane-wave propagation. However, because the electromagnetic field is coupled to charged particles, we should expect to find at a later time that a structure representing an electron or muon will have incorporated within its own wave function certain terms or factors that are photonlike in character, though the structure as a whole is at rest.

The photonlike features within a charged structure would include the Coulomb field considered as an attached photon, and the intrinsic spin magnetic field, similarly considered as an attached photon, or as an aspect of the attached photon that accounts for the Coulomb field.

We will use a wave equation of the form (6.2) for each of the two ξ quanta comprising the photon structure. For the present analysis, we can neglect the triple-exchange term as being too miniscule to be incorporated. We will also omit the pair-exchange term. The rationale for doing this is primarily that we will be using ξ waves that are designed to float on the vacuum sea, and we are using the sea in its flat canonical form. As for pair exchanges involving only the two quanta in the inner photon structure, we will anticipate that the requirement of antisymmetry within the structure will embody any consequences of these pair exchanges without the need to include the pair-exchange interaction term explicitly in the wave equation for each quantum.

We are left with two ostensibly uncoupled wave equations to be satisfied by the wave function for the pair of ξ quanta:

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_1}+\frac{1}{i}\tau_{1\xi}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\right]\Psi(1,2) \tag{7.1a}$$

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_2}+\frac{1}{i}\tau_{2\xi}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_2\right]\Psi(1,2) \tag{7.1b}$$

We here introduce centroid coordinates,

$$\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2 \tag{7.2a}$$

$$T=(t_1+t_2)/2 \tag{7.2b}$$

and relative coordinates,

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2 \tag{7.3a}$$

$$t_r=t_1-t_2 \tag{7.3b}$$

The operators in (7.1) then are transformed by

$$\nabla_1 = \frac{1}{2}\nabla_R + \nabla_r \tag{7.4a}$$

$$\nabla_2 = \frac{1}{2}\nabla_R - \nabla_r \tag{7.4b}$$

$$\frac{\partial}{\partial t_1} = \frac{1}{2}\frac{\partial}{\partial T} + \frac{\partial}{\partial t_r} \tag{7.5a}$$

$$\frac{\partial}{\partial t_2} = \frac{1}{2}\frac{\partial}{\partial T} - \frac{\partial}{\partial t_r} \tag{7.5b}$$

We can now replace the wave equations (7.1) by

$$0 = \left[\frac{1}{ic}\frac{\partial}{\partial T} + \frac{1}{2i}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\nabla_R + \frac{1}{i}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\nabla_r\right]\Psi(1,2) \tag{7.6a}$$

$$0 = \left[\frac{1}{ic}\frac{\partial}{\partial t_r} + \frac{1}{4i}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\nabla_R + \frac{1}{2i}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\nabla_r\right]\Psi(1,2) \tag{7.6b}$$

We will want to write the wave function $\Psi(1,2)$ as an expansion in a set of orthogonal basis functions, and then to solve for the coefficients in that expansion. We will here let the centroid dependence be a plane wave, with the propagation four-vector $(\mathbf{K}, K_0)$, and let the relative-time dependence be through a power series in the relative-time variable t_r. In the expansion, there will be at least one term that does not vanish when we set $t_r = 0$ and $\mathbf{K} = 0$. For such a term, we can undertake to satisfy the Fermi-sea boundary condition by requiring

$$(\nabla_1^2 + \nabla_2^2)\psi_1 = -2\kappa^2\psi_1 \tag{7.7}$$

where ψ_1 is the particular term in the general expansion. This procedure allocates $-\kappa^2$ to each ∇_i^2, in accordance with (5.1) and (5.2b), and assumes that any correlation between the direction of $\nabla_1\psi_1$ and the direction of $\nabla_2\psi_1$ will be handled later through correlations among the ψ_n.

From (7.4) we can see that, in the general case,

$$\nabla_1^2 + \nabla_2^2 = \tfrac{1}{2}\nabla_R^2 + 2\nabla_r^2 \tag{7.8}$$

We would like to make the Fermi-sea boundary condition apply separately to each term in the expansion, and thus be independent of the centroid momentum **K**. That is, we would like to make this boundary condition give no restriction on ∇_R^2, but only a restriction on ∇_r^2. We can accomplish this by generalizing (7.7) to take the form

$$\left(\nabla_1^2+\nabla_2^2-\tfrac{1}{2}\nabla_R^2\right)\psi_n=2\nabla_r^2\psi_n=-2\kappa^2\psi_n \tag{7.9a}$$

This can be shortened to

$$\nabla_r^2\psi_n=-\kappa^2\psi_n \tag{7.9b}$$

where ψ_n is any one of the functions in the set of basis functions in terms of which the general wave function $\Psi(1,2)$ is to be expanded.

The generalization of this procedure to structures of three or more primitive ξ quanta is obvious. In particular, we can expect the three-quantum wave function to be expanded in basis functions ψ_n that satisfy

$$\left(\nabla_1^2+\nabla_2^2+\nabla_3^2-\tfrac{1}{3}\nabla_R^2\right)\psi_n=-3\kappa^2\psi_n \tag{7.9c}$$

8. SEPARATION OF VARIABLES

The imposition of the momentum-space boundary condition (7.9b) has the effect of coupling together the wave equations (7.1a) and (7.1b). The nature of the solutions permitted by this coupling will emerge as we examine the detailed expansion of the wave function $\Psi(1,2)$.

First, we can factor out the plane-wave centroid dependence:

$$\Psi(1,2)=\Phi(1,2)\exp(i\mathbf{K}\cdot\mathbf{R}-iK_0cT) \tag{8.1}$$

The dependence upon **R** and T is assumed to be entirely included in the exponential factor in (8.1). The function $\Phi(1,2)$ will contain the dependence upon **r** and t_r. Next comes an expansion in powers of t_r:

$$\Phi(1,2)=\phi^{(0)}(\mathbf{r})+t_r\phi^{(1)}(\mathbf{r})+t_r^2\phi^{(2)}(\mathbf{r})+\cdots \tag{8.2}$$

The portion of (8.2) that is independent of t_r is then expanded into

$$\phi^{(0)}(\mathbf{r})=C_1\psi_1+C_2\psi_2+C_3\psi_3+\cdots \tag{8.3}$$

where each of the ψ_n depends upon the relative-position vector **r**, and upon the σ-spin and τ-spin parameters of the two ξ waves in the two-quantum structure.

The requirement (7.9b) which each ψ_n in (8.3) must satisfy makes it convenient to introduce as a mathematical simplification the relationship

$$\mathbf{K} = \kappa \mathbf{k} \tag{8.4a}$$

Along with this, we will introduce a similar relationship

$$K_0 = \kappa k_0 \tag{8.4b}$$

which serves to define k_0, as (8.4a) defined **k**. We expect to find that the structure representing a photon will be characterized then by

$$k_0^2 = k^2 \tag{8.5}$$

so that this structure will be in forward motion at the velocity c.

With the substitution of (8.1) and (8.4), we find that the wave equations (7.6) take the form

$$k_0 \Phi(1,2) = \left[\frac{1}{2}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \mathbf{k} + \frac{1}{i\kappa}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \nabla_r \right] \Phi(1,2) \tag{8.6a}$$

$$\frac{-1}{i\kappa c}\frac{\partial}{\partial t_r}\Phi(1,2) = \left[\frac{1}{4}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \mathbf{k} + \frac{1}{2i\kappa}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \nabla_r \right] \Phi(1,2) \tag{8.6b}$$

We can use (8.6a) to solve for (8.3), which is the initial portion of (8.2). We can then use (8.6b) to generate the higher terms in (8.2) through a Taylor expansion. In this way the complete solution can be constructed.

The most difficult part is the construction of (8.3) as a solution of (8.6a). We can write (8.6a), with $t_r = 0$, in the form

$$k_0 \phi^{(0)}(\mathbf{r}) = (H_k + H_r)\phi^{(0)}(\mathbf{r}) \tag{8.7a}$$

where

$$H_k = \frac{1}{2}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \mathbf{k} \tag{8.7b}$$

$$H_r = \frac{1}{i\kappa}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot \nabla_r \tag{8.7c}$$

We will be interested in a spin-1 solution, representing a photon, and the leading-term function in (8.3), denoted by ψ_1, will be a simplest function in which the spin-1 character is obtained from the σ spins of the two quanta, aligned parallel. The required antisymmetry is then obtained through the use of opposite τ spins for the two quanta. This leading-term function will then be characterized by

$$(\tau_{1\zeta}+\tau_{2\zeta})\psi_1=0 \tag{8.8}$$

9. SPIN FUNCTIONS

Both of the operators H_k and H_r, in (8.7), are diagonal in τ spin for the two quanta, and will therefore couple from ψ_1, satisfying (8.8), only to other functions ψ_n which satisfy

$$(\tau_{1\zeta}+\tau_{2\zeta})\psi_n=0 \tag{9.1}$$

This lets us limit the functions in the expansion of $\phi^{(0)}(\mathbf{r})$ to those functions for which the τ spins of the two quanta add to zero.

A general τ-spin function can be written as

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^{\tau} = a\chi^+(\tau_1)\chi^+(\tau_2)+b\chi^+(\tau_1)\chi^-(\tau_2)$$

$$+c\chi^-(\tau_1)\chi^+(\tau_2)+d\chi^-(\tau_1)\chi^-(\tau_2) \tag{9.2}$$

This function uses a basis system which can be represented by

$$\chi(1,2)^{\tau}=\begin{bmatrix} + & + \\ + & - \\ - & + \\ - & - \end{bmatrix}^{\tau} \tag{9.3}$$

There are two normalized τ-spin functions which are compatible with (8.8) and (9.1). These are

$${}^{s}\Gamma_1^0=2^{-1/2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^{\tau} \tag{9.4a}$$

$${}^{a}\Gamma_0^0=2^{-1/2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}^{\tau} \tag{9.4b}$$

A general σ-spin function can similarly be written as

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^{\sigma} = a\chi^{+}(\sigma_1)\chi^{+}(\sigma_2) + b\chi^{+}(\sigma_1)\chi^{-}(\sigma_2)$$

$$+ c\chi^{-}(\sigma_1)\chi^{+}(\sigma_2) + d\chi^{-}(\sigma_1)\chi^{-}(\sigma_2) \tag{9.5}$$

This function uses a basis system which parallels (9.3):

$$\chi(1,2)^{\sigma} = \begin{bmatrix} + & + \\ + & - \\ - & + \\ - & - \end{bmatrix}^{\sigma} \tag{9.6}$$

Here we find a larger set of σ-spin functions compatible with the requirements of the photon structure.

The simplest of the σ-spin functions we will use is the S-state function with $J=1$, whose three components are

$$^{3}(1)_{1}^{+1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\sigma}, \quad ^{3}(1)_{1}^{0} = 2^{-1/2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^{\sigma}, \quad ^{3}(1)_{1}^{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^{\sigma} \tag{9.7}$$

When the σ-spin function (9.7) is combined with the τ-spin function (9.4b) and a spherical Bessel function $j_0(\kappa r)$, the result is the initial function in the ψ_n expansion:

$$\psi_1 = \kappa\, {}^{a}\Gamma_0^0\, {}^{3}(1)_1\, j_0(\kappa r) \tag{9.8}$$

The factor κ makes ψ_1 dimensionally consistent with the single-quantum wave function given earlier in (5.1). Operations upon (9.8) by the operators (8.7b) and (8.7c) then generate both the remaining ψ_n and matrix representations for these operators in terms of ψ_n as a set of basis functions.

We can rewrite these two operators as

$$H_k = \frac{1}{4}(\tau_{1\zeta} + \tau_{2\zeta})(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{k} + \frac{1}{4}(\tau_{1\zeta} - \tau_{2\zeta})(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{k} \tag{9.9a}$$

$$H_r = \frac{1}{2i\kappa}(\tau_{1\zeta} + \tau_{2\zeta})(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\nabla_r + \frac{1}{2i\kappa}(\tau_{1\zeta} - \tau_{2\zeta})(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\nabla_r \tag{9.9b}$$

In each case, the first term on the right-hand side can be omitted, since it gives zero when acting upon either of the τ-spin functions (9.4), in accordance with (8.8) and (9.1). The second term, in each case, contains a factor which just interchanges the τ-spin functions (9.4), through

$$\tfrac{1}{2}(\tau_{1\zeta}-\tau_{2\zeta})\,{}^s\Gamma_1^0={}^a\Gamma_0^0 \tag{9.10a}$$

$$\tfrac{1}{2}(\tau_{1\zeta}-\tau_{2\zeta})\,{}^a\Gamma_0^0={}^s\Gamma_1^0 \tag{9.10b}$$

The remaining factor, for the case of H_k, can be written in a matrix form based on (9.6), given explicitly by

$$\frac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}$$

$$=\left|\begin{array}{cccc} 0 & -(k_x-ik_y)/2 & (k_x-ik_y)/2 & 0 \\ -(k_x+ik_y)/2 & k_z & 0 & (k_x-ik_y)/2 \\ (k_x+ik_y)/2 & 0 & -k_z & -(k_x-ik_y)/2 \\ 0 & (k_x+ik_y)/2 & -(k_x+ik_y)/2 & 0 \end{array}\right|^{\sigma} \tag{9.11}$$

A similar matrix for use in H_r is

$$\frac{1}{i\kappa}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r$$

$$=\frac{1}{i\kappa}\left|\begin{array}{cccc} 2\frac{\partial}{\partial z} & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) & 0 \\ \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & 0 & 0 & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \\ \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & 0 & 0 & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \\ 0 & \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & -2\frac{\partial}{\partial z} \end{array}\right|^{\sigma} \tag{9.12}$$

Each matrix is scalar in form, and will therefore not change the value of M_J when it operates upon a function such as ψ_1 in (9.8). Thus we can carry out calculations using only one of the three components in (9.7), such as the first one with $M_J = +1$, and we will generate the corresponding first component of each of the other ψ_n in the set.

At the present stage, accordingly, we need only specify the form for this first component, when we define other σ-spin functions. We can always generate the other two components, with $M_J = 0$ and $M_J = -1$, through the use of the standard formula

$$\varphi_J^{M-1} = \left[(J+M)(J-M+1)\right]^{-1/2}(J_x - iJ_y)\varphi_J^M \tag{9.13}$$

where J_x and J_y are components of the operator

$$\mathbf{J} = \frac{1}{i}\mathbf{r}\times\nabla_r + \frac{1}{i}\mathbf{k}\times\nabla_k + \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \tag{9.14}$$

There are twelve σ-spin functions which can participate as factors in the functions ψ_n. Six have even parity, the other six have odd parity. Nine are triplet functions like (9.7), while three are singlet functions. The three singlets are all ${}^1P$ functions; the nine triplets include one ${}^3S$, three ${}^3P$, and five ${}^3D$ functions. Appendix A gives explicit forms for these σ-spin functions, together with mnemonics by which they can be compactly identified and used in definitions and equations. Thus the ${}^3S$ function in (9.7) is abbreviated as ${}^3(1)_1$. The three ${}^1P$ functions are ${}^1(\mathbf{r})_1$, ${}^1(\mathbf{k})_1$, and ${}^1(i\mathbf{r}\times\mathbf{k})_1$, with the vector involved in their construction indicated within the parentheses. The three ${}^3P$ functions are similarly given as ${}^3(\mathbf{r})_1$, ${}^3(\mathbf{k})_1$, and ${}^3(i\mathbf{r}\times\mathbf{k})_1$. The five ${}^3D$ functions include three of even parity, ${}^3(\mathbf{rr})_1$, ${}^3(\mathbf{kk})_1$, ${}^3(\mathbf{rk}+\mathbf{kr})_1$, and two of odd parity, ${}^3(i\mathbf{rr}\times\mathbf{k})_1$ and ${}^3(i\mathbf{kr}\times\mathbf{k})_1$.

These σ-spin functions are defined to involve actual components of the vectors $\mathbf{r}(x,y,z)$ and $\mathbf{k}(k_x,k_y,k_z)$, rather than orientation angles associated with these vectors. As a consequence the accompanying radial dependence, indicated as $j_0(\kappa r)$ in (9.8), will not be the set $j_n(\kappa r)$ but a modified set which we can denote by $h_n(\kappa r)$, where

$$h_n(\kappa r) = (\kappa r)^{-n} j_n(\kappa r) \tag{9.15}$$

in order that the functions ψ_n which we obtain will each satisfy (7.9b). These functions (9.15) will satisfy the recurrence relation

$$h_n(\kappa r) = (2n+1)^{-1}\left[h_{n-1}(\kappa r) + \kappa^2 r^2 h_{n+1}(\kappa r)\right] \tag{9.16}$$

and the differentiation relation

$$\nabla_r h_n(\kappa r) = -\kappa^2 \mathbf{r} h_{n+1}(\kappa r) \tag{9.17}$$

as well as the Rodrigues formula

$$h_n(\kappa r) = (-1)^n \left[\frac{d}{\kappa^2 r\, dr} \right]^n \left[\frac{\sin(\kappa r)}{(\kappa r)} \right] \tag{9.18}$$

A general term in the ψ_n expansion will also include scalar dependence upon $(\mathbf{r}\cdot\mathbf{k})$, which will be found to be expressible in terms of Legendre polynomials.

10. σ-OPERATOR EQUATIONS

When operators such as (9.11) and (9.12) act upon the 12 σ-spin functions, they generate linear combinations of these same 12 functions. For present purposes we will need to have these linear combinations for four particular operators. For the operator $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{k}$ we find

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(1)_1 = -{}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(\mathbf{r})_1 = -\tfrac{4}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(1)_1 + {}^3(i\mathbf{r}\times\mathbf{k})_1 + {}^3(\mathbf{rk}+\mathbf{kr})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(\mathbf{k})_1 = -\tfrac{4}{3}k^2\ {}^3(1)_1 + 2\ {}^3(\mathbf{kk})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{r})_1 = 2\ {}^1(i\mathbf{r}\times\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{k})_1 = 0$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(i\mathbf{r}\times\mathbf{k})_1 = k^2\ {}^3(\mathbf{r})_1 - (\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1 + 2\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 \tag{10.1}$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{r}\times\mathbf{k})_1 = 2k^2\ {}^1(\mathbf{r})_1 - 2(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{rr})_1 = 2(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{r})_1 - \tfrac{2}{3}r^2\ {}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{kk})_1 = \tfrac{4}{3}k^2\ {}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{rk}+\mathbf{kr})_1 = 2k^2\ {}^1(\mathbf{r})_1 + \tfrac{2}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{rr}\times\mathbf{k})_1 = (\mathbf{r}\cdot\mathbf{k})\ {}^1(i\mathbf{r}\times\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 = k^2\ {}^1(i\mathbf{r}\times\mathbf{k})_1$$

The operator $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r$, acting directly on the 12 σ-spin functions, gives zero in six cases, a nonzero result in the other six:

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(1)_1=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^1(\mathbf{r})_1=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^1(\mathbf{k})_1=0$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(\mathbf{r})_1=8\ {}^3(1)_1\quad (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(\mathbf{k})_1=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^1(i\mathbf{r}\times\mathbf{k})_1=0$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(i\mathbf{r}\times\mathbf{k})_1=2\ {}^3(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(\mathbf{rr})_1=\tfrac{10}{3}\ {}^3(\mathbf{r})_1\quad (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(\mathbf{kk})_1=0 \tag{10.2}$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(\mathbf{rk}+\mathbf{kr})_1=\tfrac{10}{3}\ {}^3(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(i\mathbf{rr}\times\mathbf{k})_1=\tfrac{5}{2}\ {}^3(i\mathbf{r}\times\mathbf{k})_1+\tfrac{3}{2}\ {}^3(\mathbf{rk}+\mathbf{kr})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\nabla_r\ {}^3(i\mathbf{kr}\times\mathbf{k})_1=3\ {}^3(\mathbf{kk})_1$$

This operator, when acting on a function in the set ψ_n, can operate directly on the σ-spin functions as in (10.2), but it can also combine its action so that the gradient operator ∇_r differentiates a scalar factor such as a power of r^2 or a power of $(\mathbf{r}\cdot\mathbf{k})$, thereby generating the modified operator $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}$ or $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}$, respectively. The modified operator then acts on the σ-spin part of the ψ_n.

When the operator $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}$ acts on the 12 σ-spin functions, the results are

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(1)_1={}^3(\mathbf{r})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(\mathbf{r})_1=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(\mathbf{k})_1=0$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{r})_1=\tfrac{8}{3}r^2\ {}^3(1)_1+2\ {}^3(\mathbf{rr})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{k})_1=\tfrac{8}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(1)_1+{}^3(i\mathbf{r}\times\mathbf{k})_1+{}^3(\mathbf{rk}+\mathbf{kr})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(i\mathbf{r}\times\mathbf{k})_1=0$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{r}\times\mathbf{k})_1=-(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{r})_1+r^2\ {}^3(\mathbf{k})_1+2\ {}^3(i\mathbf{rr}\times\mathbf{k})_1 \tag{10.3}$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{rr})_1=\tfrac{2}{3}r^2\ {}^3(\mathbf{r})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{kk})_1=-\tfrac{1}{3}k^2\ {}^3(\mathbf{r})_1+(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1+2\ {}^3(i\mathbf{kr}\times\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{rk}+\mathbf{kr})_1=\tfrac{1}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{r})_1+r^2\ {}^3(\mathbf{k})_1+2\ {}^3(i\mathbf{rr}\times\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{rr}\times\mathbf{k})_1=\tfrac{1}{2}\mathbf{r}^2\ {}^3(i\mathbf{r}\times\mathbf{k})_1-(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{rr})_1+\tfrac{1}{2}r^2\ {}^3(\mathbf{rk}+\mathbf{kr})_1$$

$$(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{kr}\times\mathbf{k})_1=\tfrac{1}{2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(i\mathbf{r}\times\mathbf{k})_1+k^2\ {}^3(\mathbf{rr})_1+2r^2\ {}^3(\mathbf{kk})_1 -\tfrac{3}{2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{rk}+\mathbf{kr})_1$$

When it is the operator $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}$ that acts on the 12 σ-spin functions, the results are the following:

$$
\begin{aligned}
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(1)_1={}^3(\mathbf{k})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(\mathbf{r})_1=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(\mathbf{k})_1=0\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{r})_1=\tfrac{8}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(1)_1-{}^3(i\mathbf{r}\times\mathbf{k})_1+{}^3(\mathbf{rk}+\mathbf{kr})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{k})_1=\tfrac{8}{3}k^2\ {}^3(1)_1+2\ {}^3(\mathbf{kk})\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^1(i\mathbf{r}\times\mathbf{k})_1=0\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{r}\times\mathbf{k})_1=-k^2\ {}^3(\mathbf{r})_1+(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1+2\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 \qquad (10.4)\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{rr})_1=(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{r})_1-\tfrac{1}{3}r^2\ {}^3(\mathbf{k})_1-2\ {}^3(i\mathbf{rr}\times\mathbf{k})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{kk})_1=\tfrac{2}{3}k^2\ {}^3(\mathbf{k})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(\mathbf{rk}+\mathbf{kr})_1=k^2\ {}^3(\mathbf{r})_1+\tfrac{1}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1-2\ {}^3(i\mathbf{kr}\times\mathbf{k})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{rr}\times\mathbf{k})_1=\tfrac{1}{2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(i\mathbf{r}\times\mathbf{k})_1-2k^2\ {}^3(\mathbf{rr})_1-r^2\ {}^3(\mathbf{kk})_1\\
&\qquad\qquad+\tfrac{3}{2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{rk}+\mathbf{kr})_1\\
&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\ {}^3(i\mathbf{kr}\times\mathbf{k})_1=\tfrac{1}{2}k^2\ {}^3(i\mathbf{r}\times\mathbf{k})_1+(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{kk})_1-\tfrac{1}{2}k^2\ {}^3(\mathbf{rk}+\mathbf{kr})_1.
\end{aligned}
$$

11. MATRICES AND BASIS FUNCTIONS

The operators H_k and H_r in (9.9) will each have matrix representations in terms of the basis functions ψ_n. In addition, however, we can expect to find operator relationships that can be inferred from the operator forms directly, without requiring the explicit matrix representation to be available.

For example, we can use the operator form in (8.7b) to demonstrate that H_k satisfies

$$H_k^3=k^2H_k \qquad (11.1a)$$

$$H_k\left(H_k^2-k^2\right)=0 \qquad (11.1b)$$

The operation of H_k upon a particular basis function, such as ψ_1, repeated a few times upon the products of that operation, will be found to close in, using only a few basis functions and giving a small, finite matrix representation of the operator H_k.

When ψ_1 is the starting function, two other functions are generated. These can initially be given arbitrary normalizations, and the matrix form for H_k will then in general be unsymmetrical. Imposing a symmetry requirement upon this matrix then has the effect of fixing the normalizations of the two new functions.

If we write the initial function ψ_1 as

$$\psi_1 = \kappa h_0(\kappa r)\ {}^{a}\Gamma_0^{0\ 3}(1)_1 \tag{11.2}$$

then we find that

$$H_k \psi_1 = -\tfrac{1}{2}\psi_{2t} \tag{11.3}$$

with

$$\psi_{2t} = \kappa h_0(\kappa r)\ {}^{s}\Gamma_1^{0\ 1}(\mathbf{k})_1 \tag{11.4}$$

where the subscript t means "trial" function. A second operation by H_k gives

$$H_k \psi_{2t} = -\tfrac{2}{3}k^2\psi_1 + \psi_{3t} \tag{11.5}$$

where ψ_{3t} is defined by

$$\psi_{3t} = \kappa h_0(\kappa r)\ {}^{a}\Gamma_0^{0\ 3}(\mathbf{kk})_1 \tag{11.6}$$

Finally, a third operation by H_k gives

$$H_k \psi_{3t} = \tfrac{2}{3}k^2\psi_{2t} \tag{11.7}$$

Equations (11.3), (11.5), and (11.7) give a 3×3 matrix representation for H_k which is not symmetrical. Relative-normalization factors can be introduced into the definitions (11.4) and (11.6) which will symmetrize this representation. The normalized ψ_2 and ψ_3 are then given by

$$\psi_2 = \frac{3^{1/2}}{2k}\kappa h_0(\kappa r)\ {}^{s}\Gamma_1^{0\ 1}(\mathbf{k})_1 \tag{11.8}$$

$$\psi_3 = \frac{3\times2^{1/2}}{4k^2}\kappa h_0(\kappa r)\ {}^{a}\Gamma_0^{0\ 3}(\mathbf{kk})_1 \tag{11.9}$$

The operator H_k is then presented in symmetric form through the equations

$$H_k\psi_1 = -3^{-1/2}k\psi_2 \tag{11.10a}$$

$$H_k\psi_2 = -3^{-1/2}k\psi_1 + 2\times 6^{-1/2}k\psi_3 \tag{11.10b}$$

$$H_k\psi_3 = 2\times 6^{-1/2}k\psi_2 \tag{11.10c}$$

If we now introduce a very restricted wave function

$$\psi = C_1\psi_1 + C_2\psi_2 + C_3\psi_3 \tag{11.11}$$

and look for the solution of an eigenvalue equation

$$(H_k - w)\psi = 0 \tag{11.12}$$

then we will find that the secular equation takes the form

$$w(w^2 - k^2) = 0 \tag{11.13}$$

which is in full accordance with (11.1b).

The operator H_r in (8.7c) can be used in the same way. This operator, acting upon the initial function ψ_1, gives

$$H_r\psi_1 = \psi_{4t} \tag{11.14}$$

where

$$\psi_{4t} = i\kappa^2 h_1(\kappa r)\,{}^s\Gamma_1^{0\ 3}(\mathbf{r})_1 \tag{11.15}$$

A repeat of this operation gives

$$H_r\psi_{4t} = \tfrac{8}{3}\psi_1 - 2\psi_{8t} \tag{11.16}$$

where we have defined ψ_{8t} by

$$\psi_{8t} = \kappa^3 h_2(\kappa r)\,{}^a\Gamma_0^{0\ 3}(\mathbf{rr})_1 \tag{11.17}$$

Symmetrization gives us

$$\psi_4 = \frac{6^{1/2}}{4}\, i\kappa^2 h_1(\kappa r)\,{}^s\Gamma_1^{0\ 3}(\mathbf{r})_1 \tag{11.18}$$

Operations with H_k upon ψ_4 then generate three more functions:

$$\psi_{5t} = i\kappa^2 h_1(\kappa r)\, {}^a\Gamma_0^{0\ 1}(i\mathbf{r}\times\mathbf{k})_1 \tag{11.19}$$

$$\psi_{6t} = i\kappa^2 h_1(\kappa r)\, {}^s\Gamma_1^0\, (\mathbf{r}\cdot\mathbf{k})\, {}^3(\mathbf{k})_1 \tag{11.20}$$

$$\psi_{7t} = i\kappa^2 h_1(\kappa r)\, {}^s\Gamma_1^{0\ 3}(ik\mathbf{r}\times\mathbf{k})_1 \tag{11.21}$$

To symmetrize the resulting 4×4 matrix for H_k then requires that ψ_{6t} be augmented by the addition of a multiple of $k^2\psi_4$. After relative normalization factors have also been determined, the appropriate functions can be given explicitly as

$$\psi_5 = \frac{3\times2^{1/2}}{4k}\, i\kappa^2 h_1(\kappa r)\, {}^a\Gamma_0^{0\ 1}(i\mathbf{r}\times\mathbf{k})_1 \tag{11.22}$$

$$\psi_6 = \frac{3\times3^{1/2}}{4k^2}\, i\kappa^2 h_1(\kappa r)\, {}^s\Gamma_1^0 \left[(\mathbf{r}\cdot\mathbf{k})\, {}^3(\mathbf{k})_1 - \frac{1}{3}k^2\, {}^3(\mathbf{r})_1 \right] \tag{11.23}$$

$$\psi_7 = \frac{3}{2k^2}\, i\kappa^2 h_1(\kappa r)\, {}^s\Gamma_1^{0\ 3}(ik\mathbf{r}\times\mathbf{k})_1 \tag{11.24}$$

The operator equations analogous to (11.10) are then

$$H_k\psi_4 = 3^{-1/2}k\psi_5 \tag{11.25a}$$

$$H_k\psi_5 = 3^{-1/2}k\psi_4 - 6^{-1/2}k\psi_6 + 2^{-1/2}k\psi_7 \tag{11.25b}$$

$$H_k\psi_6 = -6^{-1/2}k\psi_5 \tag{11.25c}$$

$$H_k\psi_7 = 2^{-1/2}k\psi_5 \tag{11.25d}$$

The secular equation analogous to (11.13) is

$$w^2(w^2 - k^2) = 0 \tag{11.26}$$

which is again in accordance with (11.1b).

The operator H_r is also found to generate finite matrix representations which are useful in establishing the desired set of normalized basis functions. For example, the operator equations (11.14) and (11.16) can be completed by

$$H_r\psi_{8t} = -\tfrac{2}{3}\psi_{4t} \tag{11.27}$$

The 3×3 matrix for H_r can be symmetrized through the introduction of (11.18) and

$$\psi_8=\frac{3\times 2^{1/2}}{4}\kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^{0\ 3}(\mathbf{rr})_1 \tag{11.28}$$

The resulting operator equations are then

$$H_r\psi_1=4\times 6^{-1/2}\psi_4 \tag{11.29a}$$

$$H_r\psi_4=4\times 6^{-1/2}\psi_1-2\times 3^{-1/2}\psi_8 \tag{11.29b}$$

$$H_r\psi_8=-2\times 3^{-1/2}\psi_4 \tag{11.29c}$$

In analogy with (11.11) we can introduce a restricted wave function

$$\psi=C_1\psi_1+C_4\psi_4+C_8\psi_8 \tag{11.30}$$

and then look for the solution of an eigenvalue equation

$$(H_r-w)\psi=0 \tag{11.31}$$

The secular equation that is obtained has the form

$$w(w^2-4)=0 \tag{11.32}$$

We can also examine the operator H_r in (8.7c) directly. With the use of the usual spin operator identities, and with the use of the momentum-space boundary condition (7.9b), we find that H_r satisfies

$$H_r(H_r^2-4)=0 \tag{11.33}$$

when acting on functions in the set ψ_n. It is evident that (11.32) and (11.33) are consistent.

Repeated use of the operators H_k and H_r generates further functions in the set ψ_n and gives them appropriate relative normalization factors, through the requirement that the matrix representations obtained for H_k and H_r be symmetrical about the diagonal. Appendix B contains a listing of some of the basis functions obtained through this procedure.

12. DUAL FUNCTION SETS

The functions ψ_n listed in Appendix B are all antisymmetric with respect to the exchange of the two quanta ξ_1 and ξ_2. The simple replacement of ${}^a\Gamma_0^0$ by ${}^s\Gamma_1^0$, and of ${}^s\Gamma_1^0$ by ${}^a\Gamma_0^0$, will generate a dual set of functions all of which are symmetric with respect to this pair exchange. In the general expansion (8.2), these symmetric functions will be needed for the odd powers of the relative time t_r, in order that the overall function should be antisymmetric as required by the exclusion principle as applied to the primitive quanta. That is, functions such as $\phi^{(1)}(\mathbf{r})$ in (8.2) will need to be expanded in terms of symmetric functions.

Aside from this, we can see that the functions in Appendix B do not form a complete set of antisymmetric functions satisfying (7.9b). In particular, there is the simple antisymmetric function

$$\varphi_{1t} = \kappa h_0(\kappa r)\ {}^a\Gamma_0^0\ {}^3(\mathbf{k})_1 \tag{12.1}$$

which is as appropriate as those in Appendix B but does not appear there. Operation by H_k and H_r upon (12.1) is found to generate new functions, and repeated operations lead to many more functions. A partial listing of the normalized functions φ_n is given in Appendix C.

The normalization relative to the functions ψ_n in Appendix B has been established through the use of an operator Q_λ, which is defined by

$$Q_\lambda = \frac{1}{2i\kappa k}(\tau_{1\xi} - \tau_{2\xi})(\mathbf{k}\cdot\nabla_r) \tag{12.2}$$

This operator Q_λ commutes with the operators in the wave equations (7.6). Used as a linear operator, it couples directly between members of the set ψ_n and members of the set φ_n. Requiring that the matrix representations of Q_λ be symmetrical establishes the normalization of the set φ_n relative to the initially arbitrary normalization of the first function ψ_1 in the set ψ_n, as specified in (9.8) or (11.2).

Explicitly, we find that

$$Q_\lambda \psi_1 = k^{-1}\varphi_{3t} \tag{12.3a}$$

$$Q_\lambda \varphi_{3t} = (k/3)\psi_1 - k^{-1}\psi_{10t} \tag{12.3b}$$

where

$$\varphi_{3t} = i\kappa^2 h_1(\kappa r)\ {}^s\Gamma_1^0\ (\mathbf{r}\cdot\mathbf{k})\ {}^3(1)_1 \tag{12.4}$$

$$\psi_{10t} = \kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^0 \left[(\mathbf{r}\cdot\mathbf{k})^2 - \tfrac{1}{3}r^2k^2\right] {}^3(1)_1 \tag{12.5}$$

From (12.3) we can determine the required normalization factor to be inserted in (12.4), giving

$$\varphi_3 = \frac{3^{1/2}}{k} i\kappa^2 h_1(\kappa r)\, {}^s\Gamma_1^0\, (\mathbf{r}\cdot\mathbf{k})\, {}^3(1)_1 \tag{12.6}$$

Operations with H_k and H_r in (9.9) will then generate the other functions in the set φ_n, and symmetrization of the matrix representations of H_k and H_r will give these functions the normalized form shown in the listing in Appendix C.

Interchange of ${}^s\Gamma_1^0$ and ${}^a\Gamma_0^0$ in the set φ_n will give still another dual set of functions, the symmetric functions corresponding to the antisymmetric set φ_n. The operator expression on the right-hand side of the relative-time equation (8.6b) will be found to couple directly between the antisymmetric functions ψ_n and φ_n and the functions in the associated symmetric dual sets.

13. FAMILIES OF FUNCTIONS

While the functions in the sets ψ_n and φ_n have been generated individually by repeated operations using H_k and H_r, we can see on closer inspection that they group into families for which generalized formulas can be written.

It is the dependence upon $(\mathbf{r}\cdot\mathbf{k})$ that distinguishes members within one family from each other. If we denote the cosine of the angle between the vectors $\mathbf{r}$ and $\mathbf{k}$ by the letter ζ, then we will have

$$(\mathbf{r}\cdot\mathbf{k}) = rk\zeta \tag{13.1}$$

The family of functions all containing the factor ${}^3(1)_1$ can then be written in the general form

$$\Psi_n^{(1)} = A_n^{(1)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\, r^n P_n(\zeta)\, {}^3(1)_1 \tag{13.2}$$

where

$$A_n^{(1)} = (2n+1)^{1/2} \tag{13.3}$$

$$P^\tau = \tfrac{1}{2}(\tau_{1\zeta} - \tau_{2\zeta}) \tag{13.4}$$

and $P_n(\zeta)$ are the Legendre polynomials, obtainable from

$$P_n(\zeta) = \frac{1}{2^n n!}\frac{d^n}{d\zeta^n}(\zeta^2 - 1) \tag{13.5}$$

Because of the way in which the ψ_n and φ_n were formed, a residual arbitrariness remains after the normalization procedure, in that any particular function could have been defined with an additional factor of (-1) which would remain after the matrix symmetrization. Thus the identification of the functions (13.2) among the ψ_n and φ_n will in some cases involve the insertion of a factor of (-1):

$$\psi_1=\Psi_0^{(1)}, \qquad \varphi_3=\Psi_1^{(1)}, \qquad \psi_{10}=-\Psi_2^{(1)}, \qquad \varphi_{15}=-\Psi_3^{(1)}, \qquad \psi_{22}=\Psi_4^{(1)} \tag{13.6}$$

The operator Q_λ in (12.2) has matrix elements coupling together the members of this family of functions, but none that couple this family to members of other families. We can examine the representation of Q_λ as it acts on a wave function using only this family:

$$\psi=C_1\psi_1+D_3\varphi_3+C_{10}\psi_{10}+D_{15}\varphi_{15}+C_{22}\psi_{22}+\cdots \tag{13.7}$$

and as in earlier submatrix expansions we can look for eigenfunction solutions satisfying

$$(Q_\lambda-\lambda)\psi=0 \tag{13.8}$$

When we introduce (13.7) into (13.8) and solve the resulting linear homogeneous equations for the coefficients, we find

$$D_3=3^{1/2}\lambda C_1=A_1^{(1)}P_1(\lambda)C_1 \tag{13.9a}$$

$$C_{10}=-5^{1/2}\,\tfrac{1}{2}(3\lambda^2-1)C_1=-A_2^{(1)}P_2(\lambda)C_1 \tag{13.9b}$$

$$D_{15}=-7^{1/2}\,\tfrac{1}{2}(5\lambda^3-3\lambda)C_1=-A_3^{(1)}P_3(\lambda)C_1 \tag{13.9c}$$

and so forth, with the algebraic signs in (13.9) paralleling those in (13.6).

Because (13.7) is an infinite expansion, we do not find any finite secular equation arising to give any specification on the eigenvalue λ in (13.8). Earlier we found finite secular equations from the eigenvalue equations (11.12) and (11.31), but we will see that the combined operator (H_k+H_r) gives an infinite matrix equation when used as in (8.7a). However, we will find that the joint use of infinite matrix equations for Q_λ and for (H_k+H_r) leads to a reentrant matrix equation with a finite secular equation that will give us an explicit solution for the parameter λ and the expansion coefficients C_n and D_n.

The family (13.2) is one of 12 families of antisymmetric functions into which all of the functions ψ_n and φ_n fall. The other 11 families are

$$\Psi_n^{(2)} = A_n^{(2)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^s\Gamma_1^0\left[r^{n-1}P_n'(\zeta)\,{}^1(\mathbf{r})_1 - r^n k^{-1}P_{n-1}'(\zeta)\,{}^1(\mathbf{k})_1\right] \tag{13.10}$$

$$\Psi_n^{(3)} = A_n^{(3)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^s\Gamma_1^0\left[r^n k^{-1}P_{n+1}'(\zeta)\,{}^1(\mathbf{k})_1 - r^{n-1}P_n'(\zeta)\,{}^1(\mathbf{r})_1\right] \tag{13.11}$$

$$\Psi_n^{(4)} = A_n^{(4)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^{n-1}P_n'(\zeta)\,{}^3(\mathbf{r})_1 - r^n k^{-1}P_{n-1}'(\zeta)\,{}^3(\mathbf{k})_1\right] \tag{13.12}$$

$$\Psi_n^{(5)} = A_n^{(5)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^n k^{-1}P_{n+1}'(\zeta)\,{}^3(\mathbf{k})_1 - r^{n-1}P_n'(\zeta)\,{}^3(\mathbf{r})_1\right] \tag{13.13}$$

$$\Psi_n^{(6)} = A_n^{(6)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^s\Gamma_1^0\, r^{n-1}k^{-1}P_n'(\zeta)\,{}^1(i\mathbf{r}\times\mathbf{k})_1 \tag{13.14}$$

$$\Psi_n^{(7)} = A_n^{(7)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\, r^{n-1}k^{-1}P_n'(\zeta)\,{}^3(i\mathbf{r}\times\mathbf{k})_1 \tag{13.15}$$

$$\Psi_n^{(8)} = A_n^{(8)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^{n-2}P_n''(\zeta)\,{}^3(\mathbf{rr})_1 - r^{n-1}k^{-1}P_{n-1}''(\zeta)\right.$$
$$\left.\times\,{}^3(\mathbf{rk}+\mathbf{kr})_1 + r^n k^{-2}P_{n-2}''(\zeta)\,{}^3(\mathbf{kk})_1\right] \tag{13.16}$$

$$\Psi_n^{(9)} = A_n^{(9)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^n k^{-2}P_{n+2}''(\zeta)\,{}^3(\mathbf{kk})_1 - r^{n-1}k^{-1}P_{n+1}''(\zeta)\right.$$
$$\left.\times\,{}^3(\mathbf{rk}+\mathbf{kr})_1 + r^{n-2}P_n''(\zeta)\,{}^3(\mathbf{rr})_1\right] \tag{13.17}$$

$$\Psi_n^{(10)} = A_n^{(10)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left\{\left[r^{n-1}k^{-1}P_{n+1}''(\zeta) + \frac{2n+3}{2n-1}P_{n-1}''(\zeta)\right]\right.$$
$$\left.\times\,{}^3(\mathbf{rk}+\mathbf{kr})_1 - \frac{4n+2}{2n-1}P_n''(\zeta)\cdot\left[r^{n-2}\,{}^3(\mathbf{rr})_1 + r^n k^{-2}\,{}^3(\mathbf{kk})_1\right]\right\} \tag{13.18}$$

$$\Psi_n^{(11)} = A_n^{(11)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^{n-2}k^{-1}P_n''(\zeta)\,{}^3(i\mathbf{rr}\times\mathbf{k})_1\right.$$
$$\left. - r^{n-1}k^{-2}P_{n-1}''(\zeta)\;{}^3(i\mathbf{kr}\times\mathbf{k})_1\right] \tag{13.19}$$

$$\Psi_n^{(12)} = A_n^{(12)}\kappa(i\kappa)^n h_n(\kappa r)(P^\tau)^n\, {}^a\Gamma_0^0\left[r^{n-1}k^{-2}P_{n+1}''(\zeta)\,{}^3(i\mathbf{kr}\times\mathbf{k})_1\right.$$
$$\left. - r^{n-2}k^{-1}P_n''(\zeta)\;{}^3(i\mathbf{rr}\times\mathbf{k})_1\right] \tag{13.20}$$

The normalization factors in the above equations are given by

$$A_n^{(2)}=\frac{1}{2}\left[\frac{3}{n}\right]^{1/2}, \qquad A_n^{(3)}=\frac{1}{2}\left[\frac{3}{n+1}\right]^{1/2},$$

$$A_n^{(4)}=\frac{1}{4}\left[\frac{6}{n}\right]^{1/2}, \qquad A_n^{(5)}=\frac{1}{4}\left[\frac{6}{n+1}\right]^{1/2}$$

$$A_n^{(6)}=\frac{1}{2}\left[\frac{3(2n+1)}{n(n+1)}\right]^{1/2}, \qquad A_n^{(7)}=\frac{1}{4}\left[\frac{6(2n+1)}{n(n+1)}\right]^{1/2}$$

$$A_n^{(8)}=\frac{1}{2}\left[\frac{3}{(2n-1)(n-1)n}\right]^{1/2}, \qquad A_n^{(9)}=\frac{1}{2}\left[\frac{3}{(2n+3)(n+1)(n+2)}\right]^{1/2}$$

$$A_n^{(10)}=\frac{3}{4}\left[\frac{2(2n-1)}{(2n+1)(2n+3)n(n+1)}\right]^{1/2}$$

$$A_n^{(11)}=\frac{1}{2}\left[\frac{6}{(n-1)n(n+1)}\right]^{1/2}, \qquad A_n^{(12)}=\frac{1}{2}\left[\frac{6}{n(n+1)(n+2)}\right]^{1/2} \tag{13.21}$$

Eigenfunction solutions of (13.8) can similarly be written in generalized form. The solution (13.9) can be expressed compactly as

$$C_n^{(1)}=A_n^{(1)}P_n(\lambda)C^{(1)} \tag{13.22}$$

where

$$C^{(1)}=C_1 \tag{13.23}$$

is the single adjustable coefficient that now appears, the other coefficients being determined from $C^{(1)}$ once the value of λ has been fixed. The coefficients $C_n^{(1)}$ are associated directly with the functions in the form (13.2), so that no factors of (-1) appear in (13.22). These factors only appear when the explicit functions ψ_n and φ_n are used, from Appendices B and C, together with their associated coefficients C_n and D_n.

As will be seen, the operator Q_λ couples certain of the families together, leading to eigenfunction solutions which incorporate this coupling. An example is

$$C_n^{(2)}=A_n^{(2)}\left[P_n'(\lambda)C^{(2)}-P_{n-1}'(\lambda)C^{(3)}\right] \tag{13.24a}$$

$$C_n^{(3)}=A_n^{(3)}\left[P_{n+1}'(\lambda)C^{(3)}-P_n'(\lambda)C^{(2)}\right] \tag{13.24b}$$

and another example is the similar pairing

$$C_n^{(4)} = A_n^{(4)}\left[P_n'(\lambda) C^{(4)} - P_{n-1}'(\lambda) C^{(5)} \right] \tag{13.25a}$$

$$C_n^{(5)} = A_n^{(5)}\left[P_{n+1}'(\lambda) C^{(5)} - P_n'(\lambda) C^{(4)} \right] \tag{13.25b}$$

The next two families are uncoupled,

$$C_n^{(6)} = A_n^{(6)} P_n'(\lambda) C^{(6)} \tag{13.26}$$

$$C_n^{(7)} = A_n^{(7)} P_n'(\lambda) C^{(7)} \tag{13.27}$$

but the next three are grouped together as

$$C_n^{(8)} = A_n^{(8)}\left[P_n''(\lambda) C^{(8)} - P_{n-1}''(\lambda) C^{(10)} + P_{n-2}''(\lambda) C^{(9)} \right] \tag{13.28a}$$

$$C_n^{(9)} = A_n^{(9)}\left[P_{n+2}''(\lambda) C^{(9)} - P_{n+1}''(\lambda) C^{(10)} + P_n''(\lambda) C^{(8)} \right] \tag{13.28b}$$

$$C_n^{(10)} = A_n^{(10)}\left\{ \left[P_{n+1}''(\lambda) + \frac{2n+3}{2n-1} P_{n-1}''(\lambda) \right] \cdot C^{(10)} - \frac{4n+2}{2n-1} P_n''(\lambda)\left[C^{(8)} + C^{(9)} \right] \right\} \tag{13.28c}$$

Finally, the last two families are coupled together as the pair

$$C_n^{(11)} = A_n^{(11)}\left[P_n''(\lambda) C^{(11)} - P_{n-1}''(\lambda) C^{(12)} \right] \tag{13.29a}$$

$$C_n^{(12)} = A_n^{(12)}\left[P_{n+1}''(\lambda) C^{(12)} - P_n''(\lambda) C^{(11)} \right] \tag{13.29b}$$

The 12 adjustable coefficients $C^{(j)}$ can be replaced by 12 of the C_n and D_n through (13.23) and the following:

$$\begin{aligned}
&C^{(2)} = 2\times 3^{-1/2} D_2, \quad C^{(3)} = 2\times 3^{-1/2} C_2, \quad C^{(4)} = 4\times 6^{-1/2} C_4 \\
&C^{(5)} = 4\times 6^{-1/2} D_1, \quad C^{(6)} = (2\times 2^{1/2}/3) C_5, \quad C^{(7)} = (4/3) D_4 \\
&C^{(8)} = -(2\times 2^{1/2}/3) C_8, \quad C^{(9)} = (2\times 2^{1/2}/3) C_3 \\
&C^{(10)} = (4\times 15^{1/2}/9) D_5, \quad C^{(11)} = -(2/3) D_9, \quad C^{(12)} = (2/3) C_7
\end{aligned} \tag{13.30}$$

Of the many interrelationships between individual coefficients C_n and D_n, there will be immediate use for the following five:

$$D_7 = 5^{1/2}\lambda C_3 - 2\times 6^{1/2} D_5, \qquad D_{10} = -5^{1/2}\lambda C_5$$
$$D_{13} = 2\times 6^{1/2} D_5 + 5^{1/2}\lambda C_8, \qquad C_{11} = -5^{1/2}\lambda D_4 \qquad (13.31)$$
$$C_{12} = (70^{1/2}/7)(C_3 - C_8) - (5\times 21^{1/2}/7)\lambda D_5$$

in addition to (13.9a) and (13.9b), in the reduction of the infinite secular equation to finite form, in the following section.

14. SECULAR EQUATION

We will be working here with a combined expansion of the form

$$\psi = (C_1\psi_1 + C_2\psi_2 + C_3\psi_3 + \cdots) + (D_1\varphi_1 + D_2\varphi_2 + \cdots) \qquad (14.1)$$

We will look for solutions of the wave equation

$$0 = (H - w)\psi \qquad (14.2)$$

where

$$H = H_k + H_r \qquad (14.3)$$

with the individual operators as defined in (8.7), and with $w = k_0$ being the dimensionless energy eigenvalue. For the solution to represent a photon, we will need to find the motion to be massless, so that

$$w^2 = k^2 \qquad (14.4)$$

What we have, so far, is operator equations of the form

$$H\psi_1 = -3^{-1/2}k\psi_2 + 4\times 6^{-1/2}\psi_4 \qquad (14.5)$$

obtainable from (11.10a) and (11.29a). There will be corresponding operator equations involving the φ_n, such as

$$H\varphi_1 = (2\times 2^{1/2}/3)\varphi_3 + (6^{1/2}/3)\varphi_4 + (10^{1/2}/3)\varphi_5 \qquad (14.6)$$

Substitution of (14.1) into (14.2), and the utilization of operator equations including (14.5) and (14.6) and others like them, leads to equations that interconnect the coefficients in (14.1). Because the matrix representations of H_k and H_r have been symmetrized, the matrix elements that appear in the equations for the coefficients are the same as those that appear in the equations for the functions.

Eighteen of these equations are included in Appendices D and E, nine of them relating the coefficients C_n and nine of them relating the coefficients D_n. Substitution from some of these equations into certain others leads to the following set of six equations:

$$0=\left[-w^2+(1/3)(k^2+8)\right]C_1-(2^{1/2}/3)k^2C_3+(2\times 2^{1/2}/3)kC_5 -(4\times 2^{1/2}/3)C_8 \tag{14.7a}$$

$$0=-(2^{1/2}/3)k^2C_1+\left[-w^2+(2/3)(k^2+4)\right]C_3+(2/3)kC_5 -(4\times 10^{1/2}/15)C_{10}-(2\times 10^{1/2}/5)C_{11} +(2\times 70^{1/2}/15)C_{12} \tag{14.7b}$$

$$0=(2\times 2^{1/2}/3)kC_1+(2/3)kC_3+(-w^2+k^2)C_5-(2/3)kC_8 +(2\times 10^{1/2}/15)kC_{10}+(2\times 70^{1/2}/15)kC_{12} \tag{14.7c}$$

$$0=-(4\times 2^{1/2}/3)C_1-(2/3)kC_5+\left[-w^2+(1/3)(k^2+4)\right]C_8 -(10^{1/2}/15)k^2C_{10}+(10^{1/2}/10)k^2C_{11}+(70^{1/2}/30)k^2C_{12} \tag{14.7d}$$

$$0=\left[-w^2+(1/2)(k^2+4)\right]D_4+(15^{1/2}/10)(k^2+4)D_5-(10^{1/2}/10)k^2D_7 +(2\times 10^{1/2}/5)D_{13} \tag{14.7e}$$

$$0=(2\times 5^{1/2}/15)(-k^2+4)D_3+(15^{1/2}/10)(k^2+4)D_4+\left[-w^2 +(1/30)(17k^2+52)\right]D_5+(6^{1/2}/30)k^2D_7-4kD_{10}+4D_{13} \tag{14.7f}$$

We will now require that a solution of (14.2) be at the same time a solution of the auxiliary equation (13.8). This is permissible since the operators Q_λ and H are commuting operators and can be separately and simultaneously given eigenvalues. When we introduce this requirement, we can now make use of the recursion formulas in (13.9) and (13.31). Substitution of these formulas in (14.7) brings us to a set of six linear homogeneous equations in the six coefficients C_1, C_3, C_5, C_8, D_4, and D_5. For a solution to exist, we then require that the secular determinant for these six equations be equal to zero.

The standard procedure for solving a set of linear homogeneous equations has been used here, in an algebraic form described in Appendix F. The secular equation that is obtained by this procedure is

$$0=\left[w^4-w^2(k^2+4)+4k^2\lambda^2\right]^3 \tag{14.8}$$

We are here interested in structures that satisfy the massless wave equation (14.4). By inspection we see that (14.8) can be made consistent with (14.4) only if we require that

$$\lambda^2=1 \tag{14.9}$$

With (14.9) inserted, (14.8) takes the form

$$0=\left[(w^2-k^2)(w^2-4)\right]^3 \tag{14.10}$$

Since the factor (w^2-k^2) appears cubed in (14.10), there will be three independent solutions for the set of six coefficients that will be associated with this root of the secular equation. The algebraic procedure described in Appendix *F* lets us solve for three independent solutions. One solution, in which D_5 is assumed nonzero, leads to $C_1=0$. The second solution, in which D_5 is assumed to vanish, while D_4 is nonzero, also leads to $C_1=0$. There is a third solution, in which D_5 and D_4 are both assumed to vanish, and in this case there is a nonvanishing result for C_1. A general solution will be a linear combination of these three particular solutions.

Furthermore, the requirement (14.9) leaves us the options of $\lambda=+1$ and $\lambda=-1$, each of which will give a solution consistent with (14.4).

Accordingly, we are left with too many solutions. We can narrow the selection to some extent by introducing the requirement that the physical photon possess only two independent polarizations, instead of the three which would be expected from the three available choices of M_J that are given for the functions listed in Appendix *A*. That is, we can require that, when the propagation vector **k** is directed along the z axis, so that $k_x=k_y=0$, the component of the wave function with $M_J=0$, referred to the same z axis, should vanish identically. Only the two circular polarizations, $M_J=+1$, and $M_J=-1$, should remain as nonzero possibilities. This requirement can be expressed explicitly by

$$0=C_1-2^{1/2}C_3 \tag{14.11}$$

The restriction (14.11) removes two of the six solutions. We are left with four solutions. Two of them are characterized by

$$C_1 = 2^{1/2} C_3 = 1 \tag{14.12a}$$

$$D_1 = 0 \tag{14.12b}$$

with the options of $\lambda = +1$ and $\lambda = -1$ identifying the two. The other two are characterized by

$$C_1 = C_3 = 0 \tag{14.13a}$$

$$D_1 = -\lambda 6^{1/2}/2 \tag{14.13b}$$

with again the two options of $\lambda = +1$ and $\lambda = -1$.

Each of these solutions is a mixture of two polarizations. While we have removed the longitudinal polarization through the specification (14.11), we have not yet clearly distinguished two solutions which individually represent the two circular polarizations that a photon can have.

At the center of the structure, specified by $\mathbf{r} = 0$, all of the expansion functions vanish except for those that contain the radial function $h_0(\kappa r)$. These are the four functions ψ_1, ψ_2, ψ_3, and φ_1, but the condition (14.11) ensures that ψ_2 does not appear in our solutions. The remaining three functions are those whose coefficients appear in (14.12) and (14.13). The σ-spin functions contained in these three can be combined to give the following composite σ-spin function:

$$^3\alpha_1 = \frac{1}{3}\,{}^3(1)_1 - \frac{\lambda}{4k}\,{}^3(\mathbf{k})_1 + \frac{1}{4k^2}\,{}^3(\mathbf{kk})_1 \tag{14.14}$$

When the propagation vector $\mathbf{k}$ is along the positive z axis, so that

$$k_x = k_y = 0, \qquad k_z > 0 \tag{14.15a}$$

we find that

$$^3\alpha_1^{+1} = {}^3(1)_1^{+1}, \qquad {}^3\alpha_1^0 = {}^3\alpha_1^{-1} = 0 \qquad \text{for } \lambda = -1 \tag{14.15b}$$

$$^3\alpha_1^{+1} = {}^3\alpha_1^0 = 0, \qquad {}^3\alpha_1^{-1} = {}^3(1)_1^{-1} \qquad \text{for } \lambda = +1 \tag{14.15c}$$

That is, when we use the grouping (14.14), we have a composite σ-spin function with $\mathbf{J}$ parallel to $\mathbf{k}$ for $\lambda = -1$, but with $\mathbf{J}$ antiparallel to $\mathbf{k}$ for $\lambda = +1$. We can make this simplicity apply to the innermost part of the structure if we add one-third of the solution (14.12) to one-third of the solution (14.13). The resulting composite solution is then characterized by

$$C_1 = 1/3, \qquad C_3 = 2^{1/2}/6, \qquad D_1 = -\lambda 6^{-1/2} \tag{14.16}$$

The complete set of C_n and D_n can be determined from the following 12 leading coefficients:

$$\begin{aligned}
&C^{(1)} = 1/3, \quad C^{(2)} = -\lambda(k+2)/3, \quad C^{(3)} = 0, \quad C^{(4)} = k/3 \\
&C^{(5)} = -2\lambda/3, \quad C^{(6)} = (k+2)/3, \quad C^{(7)} = -\lambda(k+2)/3, \\
&C^{(8)} = -(3k+4)/9 \quad C^{(9)} = 2/9, \quad C^{(10)} = -\lambda(3k+2)/9, \\
&C^{(11)} = \lambda(k+2)/6, \quad C^{(12)} = (k+2)/6
\end{aligned} \tag{14.17}$$

Some of the C_n are

$$\begin{aligned}
&C_1 = 1/3, \quad C_2 = 0, \quad C_3 = 2^{1/2}/6, \quad C_4 = 6^{1/2}k/12 \\
&C_5 = 2^{1/2}(k+2)/4, \quad C_6 = -3^{1/2}(k+6)/12, \quad C_7 = (k+2)/4, \\
&C_8 = 2^{1/2}(3k+4)/12, \quad C_9 = 6^{1/2}(k+2)/4, \quad C_{10} = -5^{1/2}/3 \\
&C_{11} = 5^{1/2}(k+2)/4, \quad C_{12} = 35^{1/2}(3k+2)/84
\end{aligned} \tag{14.18}$$

Some of the D_n are

$$\begin{aligned}
&D_1 = -\lambda 6^{1/2}/6, \quad D_2 = -\lambda 3^{1/2}(k+2)/6, \quad D_3 = \lambda 3^{1/2}/3 \\
&D_4 = -\lambda(k+2)/4, \quad D_5 = -\lambda 15^{1/2}(3k+2)/60, \quad D_6 = \lambda 6^{1/2}(k+2)/12 \\
&D_7 = \lambda 10^{1/2}(k+4)/20, \quad D_8 = -\lambda 3^{1/2}(3k+2)/12, \quad D_9 = -\lambda(k+2)/4 \\
&D_{10} = -\lambda 10^{1/2}(k+2)/4, \quad D_{11} = \lambda 2^{1/2}(k+4)/4, \quad D_{12} = -\lambda(k+2)/2
\end{aligned} \tag{14.19}$$

Detailed examination shows that the outer parts of the structure behave similarly to the innermost part. **J** is parallel to **k** for $\lambda = -1$, antiparallel for $\lambda = +1$.

15. SOLUTION IN CLOSED FORM

The solution obtained in the previous section is embodied in the coefficients (14.17), together with the formulas that give the $C_n^{(j)}$ in terms of these leading coefficients, and the formulas that give the expansion functions $\Psi_n^{(j)}$. We will find that the expansions can in fact be summed in closed form.

We will look first at the family of functions $\Psi_n^{(1)}$ in (13.2), with $A_n^{(1)}$ given in (13.3). The summation coefficients are given in (13.22), where we now can use the result (14.9) to permit the substitution

$$P_n(\lambda) = \lambda^n \tag{15.1}$$

which is valid as long as $\lambda = +1$ or $\lambda = -1$.

The summation over the functions in this family then can be written as

$$C^{(1)}\Psi^{(1)} = \sum_n C^{(1)}\kappa(2n+1)(i\kappa\lambda P^\tau)^n r^n h_n(\kappa r) P_n(\zeta)\, {}^a\Gamma_0^0\, {}^3(1)_1 \tag{15.2}$$

This summation serves to define $\Psi^{(1)}$. We can now use (9.15) to replace $(\kappa r)^n h_n(\kappa r)$ by the usual spherical Bessel function $j_n(\kappa r)$. The summation in (15.2) can then be recognized as the expansion of a plane wave, which can be written in the form (Morse and Feshbach, 1953):

$$\begin{aligned} \Psi^{(0)} &= \kappa \exp[\, i(\lambda\kappa/k)(\mathbf{r}\cdot\mathbf{k})P^\tau]\, {}^a\Gamma_0^0 \\ &= \sum_n \kappa(2n+1)(i\kappa\lambda P^\tau)^n j_n(\kappa r) P_n(\zeta)\, {}^a\Gamma_0^0 \end{aligned} \tag{15.3}$$

We can now write

$$\Psi^{(1)} = {}^3(1)_1\, \Psi^{(0)} \tag{15.4}$$

The other families can similarly be summed into closed form. In carrying out these summations, we will need identities obtainable from (15.3) through differentiation with respect to ζ, using (13.1). We will also need to use

$$P_n'(\pm 1) = \tfrac{1}{2} n(n+1)(\pm 1)^{n+1} \tag{15.5a}$$

$$P_n''(\pm 1) = \tfrac{1}{8}(n-1)n(n+1)(n+2)(\pm 1)^n \tag{15.5b}$$

and the usual recursion and differentiation formulas for Legendre polynomials.

The solution, as obtained thus far, will be written as

$$\sum_n C_n^{(1)}\Psi_n^{(1)} + \sum_n C_n^{(2)}\Psi_n^{(2)} + \cdots + \sum_n C_n^{(12)}\Psi_n^{(12)}$$
$$= C^{(1)}\Psi^{(1)} + C^{(2)}\Psi^{(2)} + \cdots + C^{(12)}\Psi^{(12)} \tag{15.6}$$

The leading coefficients $C^{(j)}$ are given in (14.17). The closed-form functions $\Psi^{(j)}$ are given in (15.4) and in the following listing:

$$\Psi^{(2)} = \frac{3i\kappa}{8}\left[{}^1(\mathbf{r})_1 - \frac{1}{k^2}(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{k})_1 \right]\Psi^{(0)} \tag{15.7}$$

$$\Psi^{(3)} = \frac{3}{4}P^{\tau}\left[\frac{1}{k}\ {}^1(\mathbf{k})_1 \right]\Psi^{(0)} - \lambda\Psi^{(2)} \tag{15.8}$$

$$\Psi^{(4)} = \frac{3i\kappa}{16}P^{\tau}\left[{}^3(\mathbf{r})_1 - \frac{1}{k^2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1 \right]\Psi^{(0)} \tag{15.9}$$

$$\Psi^{(5)} = \frac{3}{8}\left[\frac{1}{k}\ {}^3(\mathbf{k})_1 \right]\Psi^{(0)} - \lambda\Psi^{(4)} \tag{15.10}$$

$$\Psi^{(6)} = \frac{3i\kappa}{8}\left[\frac{1}{k}\ {}^1(i\mathbf{r}\times\mathbf{k})_1 \right]\Psi^{(0)} \tag{15.11}$$

$$\Psi^{(7)} = \frac{3i\kappa}{16}P^{\tau}\left[\frac{1}{k}\ {}^3(i\mathbf{r}\times\mathbf{k})_1 \right]\Psi^{(0)} \tag{15.12}$$

$$\Psi^{(8)} = \frac{3}{16}(i\kappa)^2\left[{}^3(\mathbf{rr})_1 - \frac{1}{k^2}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{rk}+\mathbf{kr})_1 + \frac{r^2}{2k^2}\ {}^3(\mathbf{kk})_1 \right.$$
$$\left. + \frac{1}{2k^4}(\mathbf{r}\cdot\mathbf{k})^2\ {}^3(\mathbf{kk})_1 \right]\Psi^{(0)} \tag{15.13}$$

$$\Psi^{(9)} = \frac{9}{8}\left[\frac{1}{k^2}\ {}^3(\mathbf{kk})_1 \right]\Psi^{(0)} - \Psi^{(8)} - 2\lambda\Psi^{(10)} \tag{15.14}$$

$$\Psi^{(10)} = \frac{3i\kappa}{16}P^{\tau}\left[\frac{1}{k}\ {}^3(\mathbf{rk}+\mathbf{kr})_1 - \frac{2}{k^3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{kk})_1 \right]\Psi^{(0)} - \lambda\Psi^{(8)} \tag{15.15}$$

$$\Psi^{(11)} = \frac{3}{16}(i\kappa)^2\left[\frac{1}{k}\ {}^3(i\mathbf{rr}\times\mathbf{k})_1 - \frac{1}{k^3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 \right]\Psi^{(0)} \tag{15.16}$$

$$\Psi^{(12)} = \frac{3i\kappa}{4}P^{\tau}\left[\frac{1}{k^2}\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 \right]\Psi^{(0)} - \lambda\Psi^{(11)} \tag{15.17}$$

16. DEPENDENCE ON RELATIVE TIME

The analysis thus far has used the centroid wave equation (7.6a) but has not made use of the relative-time wave equation (7.6b), given also in the form (8.6b).

In analogy with the operators in (8.7), we can define relative-time operators

$$H_{kt} = \frac{1}{4}(\tau_{1\zeta}\boldsymbol{\sigma}_1 - \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\mathbf{k} \tag{16.1a}$$

$$H_{rt} = \frac{1}{2i\kappa}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r \tag{16.1b}$$

We can use the spin operators in (8.7) and (16.1), together with the boundary condition (7.9b), to establish the following identity:

$$(H_k + H_r)^2 + 4(H_{kt} + H_{rt})^2 = k^2 + 4 \tag{16.2}$$

We can see immediately that, for the massless structure satisfying (14.4), we should expect to find

$$(H_t)^2 = (H_{kt} + H_{rt})^2 = 1 \tag{16.3}$$

Since all terms in the expansion satisfy τ-spin identities of the form (9.1), we can rewrite (16.1) as

$$H_{kt} = \frac{1}{4}P^{\tau}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{k} \tag{16.4a}$$

$$H_{rt} = \frac{1}{2i\kappa}P^{\tau}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r \tag{16.4b}$$

where P^{τ} is the exchange operator (13.4). In using these operators, we will need the relationships (10.1) and (10.4). In addition, we will need the differentiation identities

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(1)_1 = 0 \qquad (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^1(\mathbf{r})_1 = -4\, {}^3(1)_1$$

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^1(\mathbf{k})_1 = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(\mathbf{r})_1 = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(\mathbf{k})_1 = 0$$

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^1(i\mathbf{r}\times\mathbf{k})_1 = -2\, {}^3(\mathbf{k})_1 \qquad (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(i\mathbf{r}\times\mathbf{k})_1 = -4\, {}^1(\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(\mathbf{rr})_1 = \frac{20}{3}\, {}^1(\mathbf{r})_1 \qquad (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(\mathbf{kk})_1 = 0 \tag{16.5}$$

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(\mathbf{rk}+\mathbf{kr})_1 = \frac{20}{3}\, {}^1(\mathbf{k})_1 \qquad (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(i\mathbf{rr}\times\mathbf{k})_1 = 5\, {}^1(i\mathbf{r}\times\mathbf{k})_1$$

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}_r\, {}^3(i\mathbf{kr}\times\mathbf{k})_1 = 0$$

and the algebraic identities

$$
\begin{aligned}
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(1)_1 &= -\ {}^1(\mathbf{r})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(\mathbf{r})_1 &= -\tfrac{4}{3}r^2\ {}^3(1)_1+2\ {}^3(\mathbf{rr})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(\mathbf{k})_1 &= \tfrac{4}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^3(1)_1-\ {}^3(i\mathbf{r}\times\mathbf{k})_1+\ {}^3(\mathbf{rk}+\mathbf{kr})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{r})_1 &= 0\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{k})_1 &= -2\ {}^1(i\mathbf{r}\times\mathbf{k})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^1(i\mathbf{r}\times\mathbf{k})_1 &= (\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{r})_1-r^2\ {}^3(\mathbf{k})_1+2\ {}^3(i\mathbf{rr}\times\mathbf{k})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{r}\times\mathbf{k})_1 &= 2(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{r})_1-2r^2\ {}^1(\mathbf{k})_1 \qquad (16.6)\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{rr})_1 &= \tfrac{4}{3}r^2\ {}^1(\mathbf{r})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{kk})_1 &= -\tfrac{2}{3}k^2\ {}^1(\mathbf{r})_1+2(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{k})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(\mathbf{rk}+\mathbf{kr})_1 &= \tfrac{2}{3}(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{r})_1+2r^2\ {}^1(\mathbf{k})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{rr}\times\mathbf{k})_1 &= r^2\ {}^1(i\mathbf{r}\times\mathbf{k})_1\\
(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 &= (\mathbf{r}\cdot\mathbf{k})\ {}^1(i\mathbf{r}\times\mathbf{k})_1
\end{aligned}
$$

Using H_t as in (16.3) for the sum of (16.1a) and (16.1b), and inserting the factor P^τ to permit us to deal only with the antisymmetric functions already constructed (as given, for example, in Appendices B and C), we obtain relationships such as the following:

$$P^\tau H_t\psi_1 = k6^{-1/2}\varphi_1 - 3^{-1/2}\varphi_2 \qquad (16.7)$$

$$P^\tau H_t\varphi_1 = k6^{-1/2}\psi_1 + k(3^{1/2}/6)\psi_3 - 3^{-1/2}\psi_5 \qquad (16.8)$$

$$P^\tau H_t\varphi_2 = -3^{-1/2}\psi_1 - (6^{1/2}/3)\psi_8 \qquad (16.9)$$

Substitution from (16.8) and (16.9) into (16.7) gives

$$(H_t)^2\psi_1 = 6^{-1}(k^2+2)\psi_1 + (2^{1/2}/12)(k^2\psi_3 - 2k\psi_5 + 4\psi_8) \qquad (16.10)$$

Because of the symmetrization of the functions, we can replace the functions ψ_n in (16.10) by the coefficients C_n and the equation will remain valid. Insertion of the explicit coefficients from (14.18) then verifies (16.3).

When we carry out a similar replacement and substitution in any of (16.7)–(16.9), we find that the operator product $(P^{\tau}H_t)$ has λ as its eigenvalue. When we now refer back to the relative-time wave equation in the form (8.6b), we find that the wave function there will satisfy the equation

$$P^{\tau}\frac{\partial}{\partial t_r}\Phi(1,2) = -i\kappa\lambda c\Phi(1,2) \tag{16.11}$$

This indicates a simple exponential dependence upon the relative time t_r.

We can easily incorporate this dependence into the closed-form expressions (15.4) and (15.7)–(15.17). For this, we generalize $\Psi^{(0)}$ in (15.3) to take the form

$$\Psi_t^{(0)} = \kappa\exp\left\{ i\kappa\lambda P^{\tau}\left[\frac{1}{k}(\mathbf{r}\cdot\mathbf{k}) - ct_r\right]\right\}\ {}^{a}\Gamma_0^0 \tag{16.12}$$

Use of (16.12) instead of (15.3) now makes the closed-form solution (15.6) satisfy the relative-time wave equation (7.6b) or (8.6b).

As we can see, (16.12) is describing a wave motion, or more correctly a pair of waves moving in opposite directions as indicated by (7.3). These directions are parallel and antiparallel to the propagation vector **k** for the structure as a whole. However, the *direction* of **k** is all that enters into (16.12); the *periodicity* of the wave motion is determined instead by the Fermi height κ. This periodicity for the pair of waves is what we need in order to satisfy the requirement, discussed in earlier sections, that the waves in the structure should merge into the surface of the vacuum sea.

17. FIELD QUANTITIES

It is only the direction of **k**, not its magnitude k, that enters into (16.12) or into any of the 12 $\Psi^{(j)}$ that are determined from (16.12) by (15.4) and (15.7)–(15.17). The magnitude k does enter, however, into the leading coefficients $C^{(j)}$, as given in (14.17). That is, the energy of the photon appears through (14.17) in the relative magnitudes of the elements forming the structure, in addition to the overall phase factor that appears in (8.1).

It is the *relative* magnitude that has been emphasized, since we have not as yet specified anything about *absolute* normalization. The expansion functions such as ψ_1 are not normalizable because they are not localized. They do not describe localized structures in a conventional sense. The structures we are discussing are standing-wave systems containing converging and diverging waves which have a central region of convergence and divergence, but which are nevertheless spread through space.

We have constructed a bilocal wave function which has some of the attributes of a photon. It satisfies the massless wave equation (14.4). It has a spin of 1, but only two polarizations rather than three.

For $\lambda = -1$, and for $k_x = k_y = 0$, with $k_z > 0$, only the components for $M_J = +1$ are nonvanishing. For $\lambda = +1$, and for $\mathbf{k}$ similarly directed along the positive z axis, only the components for $M_J = -1$ are nonvanishing.

We can consider these two choices, $\lambda = -1$ and $\lambda = +1$, to describe circular polarizations which are right-handed and left-handed, respectively.

Linear polarizations can be constructed from the two circular polarizations. One linear polarization is obtained from the addition or the two wave functions, then division by $2^{1/2}$. The other linear polarization is obtained from the subtraction of the two wave functions, then multiplication by $(-i2^{-1/2})$. The new wave function is now no longer an eigenfunction of either $\mathbf{J}$ or Q_λ, and is a superposition in which λ^2 has a definite value, but λ itself does not.

First, however, we will examine some additional properties of the circular-polarization solution (14.17). In particular, the dependence of the coefficients upon the energy eigenvalue k appears to contain some kind of information about the solution, beyond what we have already considered.

When we examine the effect of the operators $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\nabla_r$ and $P^\tau(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{k}$ upon the solution specified by (14.17), we find that a linear combination of these operators is a conserved operator with an eigenvalue. We can write the result in two equivalent forms:

$$\frac{1}{i\kappa}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\left[\nabla_r + \tfrac{1}{2}i\kappa\lambda P^\tau\mathbf{k}\right]\Psi = \lambda(k+2)\Psi \tag{17.1a}$$

$$P^\tau(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\left[\mathbf{k} + \frac{2}{i\kappa}\lambda P^\tau\nabla_r\right]\Psi = 2(k+2)\Psi \tag{17.1b}$$

Similar examination of $P^\tau(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\nabla_r$ and $(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{k}$ leads to the results

$$\frac{P^\tau}{i\kappa}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\left[\nabla_r + \tfrac{1}{2}i\kappa\lambda P^\tau\mathbf{k}\right]\Psi = 0 \tag{17.2a}$$

$$(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\left[\mathbf{k} + \frac{2}{i\kappa}\lambda P^\tau\nabla_r\right]\Psi = 0 \tag{17.2b}$$

which can also be derived from (17.1) and the two wave equations.

In (17.1a) and (17.2a), we can imagine we are looking at a transformation of the relative momentum associated with the introduction of a nonzero centroid momentum. In (17.1b) and (17.2b), we can imagine we are seeing the result of a gauge transformation of the centroid propagation vector $\mathbf{k}$, associated with the presence of some kind of vector potential.

In pursuing this suggestion of some kind of vector potential, we can first assemble the full wave function $\Psi(1,2)$, including the centroid plane-wave dependence in (8.1) and the relative-time dependence in (16.12). This full wave function can be written as

$$\Psi(1,2)=\kappa\{C^{(1)}{}^{3}(1)_1+\cdots\}\exp(i\kappa\mathbf{k}\cdot\mathbf{R}-i\kappa kcT)$$

$$\times\exp\left\{i\kappa\lambda P^{\tau}\left[\frac{1}{k}(\mathbf{r}\cdot\mathbf{k})-ct_r\right]\right\}{}^{a}\Gamma_0^0 \tag{17.3}$$

Here the summation within the first pair of braces is what we get when we combine the coefficients $C^{(j)}$ in (14.17) with the corresponding expressions from (15.4) and (15.7)–(15.17).

We will define a dimensionless Hertz vector by

$$\mathbf{\Pi}=\frac{2}{\kappa^3c}\frac{\partial}{\partial t_r}\nabla_r\Psi(1,2)=\frac{2}{i\kappa^2}\lambda P^{\tau}\nabla_r\Psi(1,2) \tag{17.4}$$

We can then define tentative electric and magnetic field vectors by

$$\mathbf{E}=\nabla_R\nabla_R\cdot\mathbf{\Pi}-\frac{1}{c^2}\frac{\partial^2}{\partial T^2}\mathbf{\Pi}=2i\lambda P^{\tau}\mathbf{k}\times(\mathbf{k}\times\nabla_r)\Psi(1,2) \tag{17.5}$$

$$\mathbf{B}=(1/c^2)\nabla_R\times\frac{\partial}{\partial T}\mathbf{\Pi}=-(2ik/c)\lambda P^{\tau}(\mathbf{k}\times\nabla_r)\Psi(1,2) \tag{17.6}$$

This Hertz vector (17.4) satisfies the vector wave equation

$$0=\nabla_R\times(\nabla_R\times\mathbf{\Pi})-\nabla_R\nabla_R\cdot\mathbf{\Pi}+\frac{1}{c^2}\frac{\partial^2}{\partial T^2}\mathbf{\Pi} \tag{17.7}$$

The field vectors (17.5) and (17.6) satisfy the vacuum Maxwell equations

$$0=\nabla_R\times\mathbf{E}+\frac{\partial}{\partial T}\mathbf{B} \tag{17.8a}$$

$$0=\nabla_R\times\mathbf{B}-\frac{1}{c^2}\frac{\partial}{\partial T}\mathbf{E} \tag{17.8b}$$

The field vectors (17.5) and (17.6) are obviously perpendicular to the propagation vector $\mathbf{k}$, as they should be in a plane-wave photon. The divergence of each vector vanishes,

$$0=\nabla_R\cdot\mathbf{E}=\nabla_R\cdot\mathbf{B} \tag{17.9}$$

as it should for the situation of a plane-wave photon propagating in a vacuum.

We need to verify the polarization characteristics of these vectors as they appear in the photon structure. For this, we first examine the summation in (17.3), which we can write as

$$
\begin{aligned}
(k_x,k_y,k_z,\lambda)_1 &= \{ C^{(1)}\,{}^3(1)_1 + \cdots \} \\
&= \Big\{ \tfrac{1}{3}\,{}^3(1)_1 - \frac{\lambda}{4k}\,{}^3(\mathbf{k})_1 + \frac{1}{4k^2}\,{}^3(\mathbf{kk})_1 + \frac{i\kappa}{8}(k+2) \\
&\quad \times \left[-\lambda\,{}^1(\mathbf{r})_1 + \frac{\lambda}{k^2}(\mathbf{r}\cdot\mathbf{k})\,{}^1(\mathbf{k})_1 + \frac{1}{k}\,{}^1(i\mathbf{r}\times\mathbf{k})_1 \right] + \frac{i\kappa P^\tau}{16}(k+2) \\
&\quad \times \Big[{}^3(\mathbf{r})_1 - \frac{1}{k^2}(\mathbf{r}\cdot\mathbf{k})\,{}^3(\mathbf{k})_1 - \frac{\lambda}{k}\,{}^3(i\mathbf{r}\times\mathbf{k})_1 - \frac{\lambda}{k}\,{}^3(\mathbf{rk}+\mathbf{kr})_1 \\
&\quad + \frac{2\lambda}{k^3}(\mathbf{r}\cdot\mathbf{k})\,{}^3(\mathbf{kk})_1 + \frac{2}{k^2}\,{}^3(i\,\mathbf{kr}\times\mathbf{k})_1 \Big] \Big\}
\end{aligned}
\tag{17.10}
$$

As noted earlier, each term in (17.10) contains a σ-spin function with three possible values of M_J, but the combination is such that two of the three do not survive the summation when $\mathbf{k}$ is parallel to the z axis. Substitution from Appendix A gives

$$
(0,0,k,-1)_1^{+1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\sigma + \frac{i\kappa}{4}(k+2)(x+iy) \begin{bmatrix} 0 \\ (P^\tau+1) \\ (P^\tau-1) \\ 0 \end{bmatrix}^\sigma \tag{17.11a}
$$

$$
(0,0,k,+1)_1^{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\sigma + \frac{i\kappa}{4}(k+2)(x-iy) \begin{bmatrix} 0 \\ (P^\tau+1) \\ (P^\tau-1) \\ 0 \end{bmatrix}^\sigma \tag{17.11b}
$$

The other components vanish:

$$
(0,0,k,-1)_1^0 = (0,0,k,-1)_1^{-1} = (0,0,k,+1)_1^{+1} = (0,0,k,+1)_1^0 = 0 \tag{17.11c}
$$

When the operator ∇_r in (17.5) and (17.6) acts upon the expressions (17.11a) and (17.11b), the first part of each expression is removed because it is a constant. If we write

$$
\nabla_r = \mathbf{n}_x \frac{\partial}{\partial x} + \mathbf{n}_y \frac{\partial}{\partial y} + \mathbf{n}_z \frac{\partial}{\partial z} \tag{17.12}
$$

$$
\mathbf{k} = \mathbf{n}_z k \tag{17.13}
$$

then we find that

$$(\mathbf{k}\times\nabla_r)(x+iy)=k(\mathbf{n}_y-i\mathbf{n}_x) \tag{17.14a}$$

$$(\mathbf{k}\times\nabla_r)(x-iy)=k(\mathbf{n}_y+i\mathbf{n}_x) \tag{17.14b}$$

Adding these together will remove $\mathbf{n}_x$ and leave only $2k\mathbf{n}_y$.

We would like to be able to add the two circular-polarization wave functions and obtain a linear-polarization wave function. However, (17.3) shows that the change from $\lambda=-1$ to $\lambda=+1$ changes the sign in the relative-motion phase factor. Only if we move to the limit of

$$\mathbf{r}=0, \qquad t_r=0 \tag{17.15}$$

will we find that the magnetic field vector **B** is entirely parallel to the y axis when the two circular-polarization wave functions are added, and is entirely parallel to the x axis when they are subtracted.

We can also look at

$$\mathbf{k}\times(\mathbf{k}\times\nabla_r)(x+iy)=-k^2(\mathbf{n}_x+i\mathbf{n}_y) \tag{17.16a}$$

$$\mathbf{k}\times(\mathbf{k}\times\nabla_r)(x-iy)=-k^2(\mathbf{n}_x-i\mathbf{n}_y) \tag{17.16b}$$

Here, addition of the two circular polarizations, with (17.15), puts the electric field vector **E** entirely parallel to the x axis, while subtraction leaves **E** parallel to the y axis.

Furthermore, it can be seen from detailed substitutions that $\mathbf{E}\times\mathbf{B}$ is directed along **k** for either of these two linear polarizations. And, because of the cross products in (17.5) and (17.6), the part of the Hertz vector (17.4) associated with the action of ∇_r on the second phase factor in (17.3) does not contribute to the field vectors **E** and **B**; it is only the differentiation of **r** as it appears in (17.10) that enters into **E** and **B**.

Except for an arbitrariness with respect to units and normalization factors, we have reproduced the usual electromagnetic field vectors and field equations from this bilocal photon theory, to the extent that we can do this with no charged particles present. We will need to return to these field equations at a later point, after structures representing charged particles have been built and analyzed.

18. SUMMARY AND PREVIEW

The motivation for this program has primarily been the need to understand the existence of the muon and electron, so similar in many ways and yet so different in their rest masses. Nonlocal structures representing leptons will be discussed in subsequent work, and the presentation of the bilocal analysis has purposely been intended as a trial run for the more complex structures. The approach taken was one that could be generalized.

The initial postulate was that there exists a single primitive field, from which all particles and forces can be constructed. From its singleness a number of simple properties were inferred. The primitive quantum should be a massless fermion with a spin of one-half. Interactions should be via spin exchanges, involving σ spin and τ spin. The vacuum should be an open Fermi sea with a height specified by a wave number κ.

The Fermi height κ will vary through space, and be a measure of the gravitational potential, but this aspect was discussed only briefly here. Of more direct importance to this analysis was the role of the open Fermi sea as a boundary condition on structures to represent elementary particles.

Elementary particles are represented as standing-wave systems which float on the open Fermi sea. Outer portions of the structures are waves at the surface of the open Fermi sea. This gives a phase-space boundary condition on the structures, and introduces a common unit for the elementary particle mass spectrum, this unit being the Fermi height κ.

Since the photon is massless, the magnitude of this mass unit is not important here. However, it does determine certain features of the bilocal structure.

The bilocal wave function for the photon was expanded in an infinite series of basis functions, each of which satisfied the phase-space boundary condition. This wave function was required to satisfy two wave equations, one of them being a centroid-time equation while the other was a relative-time equation.

From the centroid-time wave equation a secular determinant, infinite in size, could be constructed. The introduction of an auxiliary operator Q_λ, commuting with both the centroid Hamiltonian and the relative Hamiltonian, aided in sorting out the solutions by transforming the infinite secular determinant to a finite 6×6 determinant which could be evaluated algebraically.

Requiring that the structure be massless led to the specification of λ, the eigenvalue of Q_λ, to be either $+1$ or -1. Further limitation of the solutions led to the explicit pair of solutions representing right-handed ($\lambda=-1$) and left-handed ($\lambda=+1$) circularly polarized photons.

The complete solution, including the relative-time dependence, could be seen to match the required phase-space boundary condition. A Hertz vector, involving differentiation with respect to the relative time and space variables, was defined, and the electric and magnetic field vectors derived from this Hertz vector were found to be transverse relative to the photon propagation vector, and to satisfy the vacuum Maxwell equations.

The bilocal photon, a quantum-mechanical boson structure, thus satisfies the vacuum Maxwell equations. It fits in a framework which makes provision for other nonlocal structures. These other structures need to be built before we can be sure that the bilocal photon is a valid concept. Specifically, the nonlocal leptons need to be built, so that we can see whether the bilocal **E** and **B** within the nonlocal structures describe the known properties of charge and magnetic moment which we attribute to the electron and muon.

We need to examine the motion of this bilocal photon structure, appropriately modified, in a region of space where there is a gravitational acceleration, expressed through a gradient in the Fermi height κ. We also need to compare the electric and magnetic fields, as obtained from (17.5) and (17.6), with terms in the expansion of the triple-exchange interaction portion of (6.2).

These problems will be addressed in subsequent work. In particular, the analogous trilocal structure has been studied in considerable detail. Requiring that the structure should move as a particle, satisfying

$$k_0^2 = k^2 + m^2 \tag{18.1}$$

where m is the rest mass in units of κ, leads to restrictions on the structure which aid in giving an explicit set of coefficients for the expansion of the wave function. Other restrictions follow from helicity specifications, and from the requirement that the wave function be an eigenfunction of two relative-time equations as well as of the centroid-time wave equation. A trilocal solution has been found with the rest mass

$$m = 1.22993283166 \tag{18.2}$$

in units of κ. The tentative identification of this solution as a muon then gives an estimated magnitude for κ. In energy units, κ then has the magnitude

$$\kappa = 85.9067 \text{ Mev} \tag{18.3}$$

while in units of inverse length κ has the magnitude already given in (6.4).

A search for other solutions to the trilocal equation system is under way. These should include the electron and the two kinds of neutrinos. The tau lepton, however, cannot be a trilocal structure in this theory, since there is an upper limit of 3κ for the rest mass obtainable in a trilocal structure, and the tau lepton is much more massive than this limit. Its mass is very close to

$$21\kappa = 1804.04 \text{ Mev} \tag{18.4}$$

which is approximately the mass that might be expected from a structure formed from the trilocal muon through the expansion of the pair of quanta which are already in a D state, primarily, into a complete D shell of 20 quanta.

A complete D shell, with five orbital states each occupied by four quanta, could be transparent in a scattering experiment, leaving only the 21st quantum to be involved in interactions with some other structure. The complete D shell could also provide a measure of stability or metastability to the structure, and therefore an anomalously long lifetime against decay.

Structures consisting only of closed shells could have particularly long lifetimes against decay. In this connection, it can be noted that the SPD combination of the three innermost complete shells should have a rest mass very close to

$$36\kappa = 3092.64 \text{ Mev} \tag{18.5}$$

which is approximately the mass of the metastable J/ψ particle. Similarly, the FGH combination of the next three complete shells should have a mass close to

$$108\kappa = 9277.92 \text{ Mev} \tag{18.6}$$

which is approximately the mass of the upsilon particle.

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APPENDIX A. σ-SPIN FUNCTIONS

$$^3(1)_1^{+1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\sigma}, \quad ^3(1)_1^{0} = 2^{-1/2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^{\sigma}, \quad ^3(1)_1^{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^{\sigma}$$

$$ {}^{1}(\mathbf{r})_1^{+1} = \begin{bmatrix} 0 \\ (x+iy) \\ -(x+iy) \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{1}(\mathbf{r})_1^{0} = 2^{1/2} \begin{bmatrix} 0 \\ -z \\ z \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{1}(\mathbf{r})_1^{-1} = \begin{bmatrix} 0 \\ -(x-iy) \\ (x-iy) \\ 0 \end{bmatrix}^{\sigma} $$

$$ {}^{1}(\mathbf{k})_1^{+1} = \begin{bmatrix} 0 \\ (k_x+ik_y) \\ -(k_x+ik_y) \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{1}(\mathbf{k})_1^{0} = 2^{1/2} \begin{bmatrix} 0 \\ -k_z \\ k_z \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{1}(\mathbf{k})_1^{-1} = \begin{bmatrix} 0 \\ -(k_x-ik_y) \\ (k_x-ik_y) \\ 0 \end{bmatrix}^{\sigma} $$

$$ {}^{3}(\mathbf{r})_1^{+1} = \begin{bmatrix} 2z \\ (x+iy) \\ (x+iy) \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{3}(\mathbf{r})_1^{0} = 2^{1/2} \begin{bmatrix} (x-iy) \\ 0 \\ 0 \\ (x+iy) \end{bmatrix}^{\sigma}, \quad {}^{3}(\mathbf{r})_1^{-1} = \begin{bmatrix} 0 \\ (x-iy) \\ (x-iy) \\ -2z \end{bmatrix}^{\sigma} $$

$$ {}^{3}(\mathbf{k})_1^{+1} = \begin{bmatrix} 2k_z \\ (k_x+ik_y) \\ (k_x+ik_y) \\ 0 \end{bmatrix}^{\sigma}, \quad {}^{3}(\mathbf{k})_1^{0} = 2^{1/2} \begin{bmatrix} (k_x-ik_y) \\ 0 \\ 0 \\ (k_x+ik_y) \end{bmatrix}^{\sigma}, \quad {}^{3}(\mathbf{k})_1^{-1} = \begin{bmatrix} 0 \\ (k_x-ik_y) \\ (k_x-ik_y) \\ -2k_z \end{bmatrix}^{\sigma} $$

$$ {}^{1}(i\mathbf{r}\times\mathbf{k})_1^{+1} = \begin{bmatrix} 0 \\ (x+iy)k_z - z(k_x+ik_y) \\ -(x+iy)k_z + z(k_x+ik_y) \\ 0 \end{bmatrix}^{\sigma} $$

$$ {}^{1}(i\mathbf{r}\times\mathbf{k})_1^{0} = 2^{1/2} \begin{bmatrix} 0 \\ -i(xk_y-yk_x) \\ i(xk_y-yk_x) \\ 0 \end{bmatrix}^{\sigma} $$

$$ {}^{1}(i\mathbf{r}\times\mathbf{k})_1^{-1} = \begin{bmatrix} 0 \\ (x-iy)k_z - z(k_x-ik_y) \\ -(x-iy)k_z + z(k_x-ik_y) \\ 0 \end{bmatrix}^{\sigma} $$

$$ {}^{3}(i\mathbf{r}\times\mathbf{k})_1^{+1} = \boxed{\begin{matrix} 2i(xk_y - yk_x) \\ (x+iy)k_z - z(k_x+ik_y) \\ (x+iy)k_z - z(k_x+ik_y) \\ 0 \end{matrix}}^{\sigma} $$

$$ {}^{3}(i\mathbf{r}\times\mathbf{k})_1^{0} = 2^{1/2}\boxed{\begin{matrix} -(x-iy)k_z + z(k_x-ik_y) \\ 0 \\ 0 \\ (x+iy)k_z - z(k_x+ik_y) \end{matrix}}^{\sigma} $$

$$ {}^{3}(i\mathbf{r}\times\mathbf{k})_1^{-1} = \boxed{\begin{matrix} 0 \\ -(x-iy)k_z + z(k_x-ik_y) \\ -(x-iy)k_z + z(k_x-ik_y) \\ -2i(xk_y - yk_x) \end{matrix}}^{\sigma} $$

$$ {}^{3}(\mathbf{rr})_1^{+1} = \boxed{\begin{matrix} z^2 - r^2/3 \\ z(x+iy) \\ z(x+iy) \\ (x+iy)^2 \end{matrix}}^{\sigma}, \qquad {}^{3}(\mathbf{rr})_1^{0} = 2^{1/2}\boxed{\begin{matrix} z(x-iy) \\ -z^2 + r^2/3 \\ -z^2 + r^2/3 \\ -z(x+iy) \end{matrix}}^{\sigma} $$

$$ {}^{3}(\mathbf{rr})_1^{-1} = \boxed{\begin{matrix} (x-iy)^2 \\ -z(x-iy) \\ -z(x-iy) \\ z^2 - r^2/3 \end{matrix}}^{\sigma}, \qquad {}^{3}(\mathbf{kk})_1^{+1} = \boxed{\begin{matrix} k_z^2 - k^2/3 \\ k_z(k_x+ik_y) \\ k_z(k_x+ik_y) \\ (k_x+ik_y)^2 \end{matrix}}^{\sigma} $$

$$ {}^{3}(\mathbf{kk})_1^{0} = 2^{1/2}\boxed{\begin{matrix} k_z(k_x-ik_y) \\ -k_z^2 + k^2/3 \\ -k_z^2 + k^2/3 \\ -k_z(k_x+ik_y) \end{matrix}}^{\sigma}, \qquad {}^{3}(\mathbf{kk})_1^{-1} = \boxed{\begin{matrix} (k_x-ik_y)^2 \\ -k_z(k_x-ik_y) \\ -k_z(k_x-ik_y) \\ k_z^2 - k^2/3 \end{matrix}}^{\sigma} $$

$$^3(\mathbf{rk}+\mathbf{kr})_1^{+1} = \boxed{\begin{matrix} 2(zk_z-\mathbf{r}\cdot\mathbf{k}/3) \\ (x+iy)k_z+z(k_x+ik_y) \\ (x+iy)k_z+z(k_x+ik_y) \\ 2(x+iy)(k_x+ik_y) \end{matrix}}^{\,\sigma}$$

$$^3(\mathbf{rk}+\mathbf{kr})_1^{0} = 2^{1/2}\boxed{\begin{matrix} (x-iy)k_z+z(k_x-ik_y) \\ -2(zk_z-\mathbf{r}\cdot\mathbf{k}/3) \\ -2(zk_z-\mathbf{r}\cdot\mathbf{k}/3) \\ -(x+iy)k_z-z(k_x+ik_y) \end{matrix}}^{\,\sigma}$$

$$^3(\mathbf{rk}+\mathbf{kr})_1^{-1} = \boxed{\begin{matrix} 2(x-iy)(k_x-ik_y) \\ -(x-iy)k_z-z(k_x-ik_y) \\ -(x-iy)k_z-z(k_x-ik_y) \\ 2(zk_z-\mathbf{r}\cdot\mathbf{k}/3) \end{matrix}}^{\,\sigma}$$

$$^3(i\mathbf{r}\mathbf{r}\times\mathbf{k})_1^{+1} = \boxed{\begin{matrix} iz(xk_y-yk_x) \\ (x+iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)-(z^2-r^2/2)(k_x+ik_y) \\ (x+iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)-(z^2-r^2/2)(k_x+ik_y) \\ (x+iy)^2k_z-z(x+iy)(k_x+ik_y) \end{matrix}}^{\,\sigma}$$

$$^3(i\mathbf{r}\mathbf{r}\times\mathbf{k})_1^{0} = 2^{1/2}\boxed{\begin{matrix} -(x-iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)+(z^2-r^2/2)(k_x-ik_y) \\ -iz(xk_y-yk_x) \\ -iz(xk_y-yk_x) \\ -(x+iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)+(z^2-r^2/2)(k_x+ik_y) \end{matrix}}^{\,\sigma}$$

$$^3(i\mathbf{r}\mathbf{r}\times\mathbf{k})_1^{-1} = \boxed{\begin{matrix} -(x-iy)^2k_z+z(x-iy)(k_x-ik_y) \\ (x-iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)-(z^2-r^2/2)(k_x-ik_y) \\ (x-iy)(zk_z-\mathbf{r}\cdot\mathbf{k}/2)-(z^2-r^2/2)(k_x-ik_y) \\ iz(xk_y-yk_x) \end{matrix}}^{\,\sigma}$$

$$^3(i\mathbf{kr}\times\mathbf{k})_1^{+1}=\begin{bmatrix} ik_z(xk_y-yk_x) \\ (x+iy)(k_z^2-k^2/2)-(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x+ik_y) \\ (x+iy)(k_z^2-k^2/2)-(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x+ik_y) \\ (x+iy)k_z(k_x+ik_y)-z(k_x+ik_y)^2 \end{bmatrix}^{\sigma}$$

$$^3(i\mathbf{kr}\times\mathbf{k})_1^{0}=2^{1/2}\begin{bmatrix} -(x-iy)(k_z^2-k^2/2)+(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x-ik_y) \\ -ik_z(xk_y-yk_x) \\ -ik_z(xk_y-yk_x) \\ -(x+iy)(k_z^2-k^2/2)+(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x+ik_y) \end{bmatrix}^{\sigma}$$

$$^3(i\mathbf{kr}\times\mathbf{k})_1^{-1}=\begin{bmatrix} -(x-iy)k_z(k_x-ik_y)+z(k_x-ik_y)^2 \\ (x-iy)(k_z^2-k^2/2)-(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x-ik_y) \\ (x-iy)(k_z^2-k^2/2)-(zk_z-\mathbf{r}\cdot\mathbf{k}/2)(k_x-ik_y) \\ ik_z(xk_y-yk_x) \end{bmatrix}^{\sigma}$$

APPENDIX B: EXPANSION FUNCTIONS ψ_n

$$\psi_1=\kappa h_0(\kappa r)\ {}^a\Gamma_0^0\ {}^3(1)_1 \qquad \psi_2=(3^{1/2}/2k)\kappa h_0(\kappa r)\ {}^s\Gamma_1^0\ {}^1(\mathbf{k})_1$$

$$\psi_3=(3\times 2^{1/2}/4k^2)\kappa h_0(\kappa r)\ {}^a\Gamma_0^0\ {}^3(\mathbf{kk})_1$$

$$\psi_4=(6^{1/2}/4)i\kappa^2 h_1(\kappa r)\ {}^s\Gamma_1^0\ {}^3(\mathbf{r})_1 \qquad \psi_5=(3\times 2^{1/2}/4k)i\kappa^2 h_1(\kappa r)\ {}^a\Gamma_0^0\ {}^1(i\mathbf{r}\times\mathbf{k})_1$$

$$\psi_6=(3\times 3^{1/2}/4k^2)i\kappa^2 h_1(\kappa r)\ {}^s\Gamma_1^0\left[(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{k})_1-(k^2/3)\ {}^3(\mathbf{r})_1\right]$$

$$\psi_7=(3/2k^2)i\kappa^2 h_1(\kappa r)\ {}^s\Gamma_1^0\ {}^3(i\mathbf{kr}\times\mathbf{k})_1 \qquad \psi_8=(3\times 2^{1/2}/4)\kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^0\ {}^3(\mathbf{rr})_1$$

$$\psi_9=(3\times 6^{1/2}/4k)\kappa^3 h_2(\kappa r)\ {}^s\Gamma_1^0\left[(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{r})_1-(r^2/3)\ {}^1(\mathbf{k})_1\right]$$

$$\psi_{10}=(3\times 5^{1/2}/2k^2)\kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^0\left[(\mathbf{r}\cdot\mathbf{k})^2-r^2k^2/3\right]\ {}^3(1)_1$$

$$\psi_{11}=(3\times 5^{1/2}/4k^2)\kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^0\ (\mathbf{r}\cdot\mathbf{k})\ {}^3(i\mathbf{r}\times\mathbf{k})_1$$

$$\psi_{12}=(9\times 35^{1/2}/28k^2)\kappa^3 h_2(\kappa r)\ {}^a\Gamma_0^0\left[(\mathbf{r}\cdot\mathbf{k})\ {}^3(\mathbf{rk}+\mathbf{kr})_1-(2k^2/3)\ {}^3(\mathbf{rr})_1 -(2r^2/3)\ {}^3(\mathbf{kk})_1\right]$$

$$\psi_{13}=(15/4k^3)\kappa^3 h_2(\kappa r)\ {}^s\Gamma_1^0\left\{\left[(\mathbf{r}\cdot\mathbf{k})^2-r^2k^2/5\right]\ {}^1(\mathbf{k})_1-(2k^2/5)(\mathbf{r}\cdot\mathbf{k})\ {}^1(\mathbf{r})_1\right\}$$

APPENDIX C: EXPANSION FUNCTIONS φ_n

$$\varphi_1=(6^{1/2}/4k)\kappa h_0(\kappa r)\ {}^{a}\Gamma_0^{0}\ {}^{3}(\mathbf{k})_1 \qquad \varphi_2=(3^{1/2}/2)i\kappa^2 h_1(\kappa r)\ {}^{a}\Gamma_0^{0}\ {}^{1}(\mathbf{r})_1$$

$$\varphi_3=(3^{1/2}/k)i\kappa^2 h_1(\kappa r)\ {}^{s}\Gamma_1^{0}\ (\mathbf{r}\cdot\mathbf{k})\ {}^{3}(1)_1 \quad \varphi_4=(3/4k)i\kappa^2 h_1(\kappa r)\ {}^{s}\Gamma_1^{0}\ {}^{3}(i\mathbf{r}\times\mathbf{k})_1$$

$$\varphi_5=(3\times 15^{1/2}/20k)i\kappa^2 h_1(\kappa r)\ {}^{s}\Gamma_1^{0}\ {}^{3}(\mathbf{rk}+\mathbf{kr})_1$$

$$\varphi_6=(3\times 6^{1/2}/4k^2)i\kappa^2 h_1(\kappa r)\ {}^{a}\Gamma_0^{0}\left[(\mathbf{r}\cdot\mathbf{k})\ {}^{1}(\mathbf{k})_1-(k^2/3)\ {}^{1}(\mathbf{r})_1\right]$$

$$\varphi_7=(3\times 10^{1/2}/4k^3)i\kappa^2 h_1(\kappa r)\ {}^{s}\Gamma_1^{0}\left[(\mathbf{r}\cdot\mathbf{k})\ {}^{3}(\mathbf{kk})_1-(k^2/5)\ {}^{3}(\mathbf{rk}+\mathbf{kr})_1\right]$$

$$\varphi_8=(3\times 3^{1/2}/4k)\kappa^3 h_2(\kappa r)\ {}^{a}\Gamma_0^{0}\left[(\mathbf{r}\cdot\mathbf{k})\ {}^{3}(\mathbf{r})_1-(r^2/3)\ {}^{3}(\mathbf{k})_1\right]$$

$$\varphi_9=(3/2k)\kappa^3 h_2(\kappa r)\ {}^{a}\Gamma_0^{0}\ {}^{3}(i\mathbf{rr}\times\mathbf{k})_1$$

$$\varphi_{10}=(3\times 10^{1/2}/4k^2)\kappa^3 h_2(\kappa r)\ {}^{s}\Gamma_1^{0}(\mathbf{r}\cdot\mathbf{k})\ {}^{1}(i\mathbf{r}\times\mathbf{k})_1$$

$$\varphi_{11}=(15\times 2^{1/2}/8k^3)\kappa^3 h_2(\kappa r)\ {}^{a}\Gamma_0^{0}\left\{\left[(\mathbf{r}\cdot\mathbf{k})^2-r^2k^2/5\right]\ {}^{3}(\mathbf{k})_1 -(2k^2/5)(\mathbf{r}\cdot\mathbf{k})\ {}^{3}(\mathbf{r})_1\right\}$$

$$\varphi_{12}=(15/4k^3)\kappa^3 h_2(\kappa r)\ {}^{a}\Gamma_0^{0}\left[(\mathbf{r}\cdot\mathbf{k})\ {}^{3}(i\mathbf{kr}\times\mathbf{k})_1-(k^2/5)\ {}^{3}(i\mathbf{rr}\times\mathbf{k})_1\right]$$

$$\varphi_{13}=(3\times 10^{1/2}/4k)i\kappa^4 h_3(\kappa r)\ {}^{s}\Gamma_1^{0}\left[(\mathbf{r}\cdot\mathbf{k})\ {}^{3}(\mathbf{rr})_1-(r^2/5)\ {}^{3}(\mathbf{rk}+\mathbf{kr})_1\right]$$

APPENDIX D: RELATIONS CONNECTING C_n

$$wC_1=-(k/3^{1/2})C_2+(4/6^{1/2})C_4 \qquad wC_2=-(k/3^{1/2})C_1+(2k/6^{1/2})C_3$$

$$wC_3=(2k/6^{1/2})C_2+(2/6^{1/2})C_6+2^{1/2}C_7$$

$$wC_4=(4/6^{1/2})C_1+(k/3^{1/2})C_5-(2/3^{1/2})C_8$$

$$wC_5=(k/3^{1/2})C_4-(k/6^{1/2})C_6+(k/2^{1/2})C_7$$

$$wC_6=(2/6^{1/2})C_3-(k/6^{1/2})C_5-(4/15^{1/2})C_{10} -(3/15^{1/2})C_{11}-(7/105^{1/2})C_{12}$$

$$wC_7=2^{1/2}C_3+(k/2^{1/2})C_5-(1/5^{1/2})C_{11}+(7/35^{1/2})C_{12}$$

$$wC_8=-(2/3^{1/2})C_4+(k/3^{1/2})C_9$$

$$wC_9=(k/3^{1/2})C_8-(2k/30^{1/2})C_{10}+(3k/30^{1/2})C_{11}+(7k/210^{1/2})C_{12}$$

APPENDIX E: RELATIONS CONNECTING D_n

$$wD_1=(2\times 2^{1/2}/3)D_3+(2/6^{1/2})D_4+(10^{1/2}/3)D_5$$

$$wD_2=-(k/3)D_3+(k/3^{1/2})D_4+(5^{1/2}k/3)D_5$$

$$wD_3=(2\times 2^{1/2}/3)D_1-(k/3)D_2-(2^{1/2}k/3)D_6-(4/3)D_8$$

$$wD_4=(2/6^{1/2})D_1+(k/3^{1/2})D_2-(k/6^{1/2})D_6+(1/3^{1/2})D_8-D_9$$

$$wD_5=(10^{1/2}/3)D_1+(5^{1/2}k/3)D_2+(10^{1/2}k/30)D_6$$
$$-(5^{1/2}/15)D_8-(3/15^{1/2})D_9$$

$$wD_6=-(2^{1/2}k/3)D_3-(k/6^{1/2})D_4+(10^{1/2}k/30)D_5+(3k/15^{1/2})D_7$$

$$wD_7=(3k/15^{1/2})D_6-(2/5^{1/2})D_{11}-(4/10^{1/2})D_{12}$$

$$wD_8=-(4/3)D_3+(1/3^{1/2})D_4-(5^{1/2}/15)D_5$$
$$+(3k/30^{1/2})D_{10}+(6/30^{1/2})D_{13}$$

$$wD_9=-D_4-(3/15^{1/2})D_5+(k/10^{1/2})D_{10}-(2/10^{1/2})D_{13}$$

APPENDIX F: ALGEBRAIC MATRIX INVERSION

In the standard method for the numerical solution of simultaneous linear equations, as described for example by Milne (1949), a step-by-step procedure is used to generate a triangular matrix from the original square matrix. This "method of elimination" is equally valid when the matrix elements contain algebraic quantities.

The procedure starts at the upper left corner of the matrix, and systematically replaces the old matrix elements by new elements. When the old matrix elements contain algebraic variables, the new elements are, in general, ratios of polynomials in these variables.

Most of the algebraic manipulations can be broken down into a series of steps in which one ratio of polynomials, X/A, is added to the product of two other ratios, Y/A and Z/B, in which one of the denominators is the same polynomial that was already present in the first ratio. When put over a common denominator, the result has the form

$$\frac{X}{A}+\frac{YZ}{AB}=\frac{1}{AB}(BX+YZ)$$

and in every instance the polynomial A will be found to be an exact algebraic factor of the new numerator. This provides a running check on the algebraic calculations, and at the same time keeps the polynomials relatively simple in form.

The solution for the six coefficients is again obtained through an algebraic analog of the method given by Milne, and again there is found to be simplification, and algebraic checking, through exact factoring of numerator expressions by polynomials that appear in the denominator.

REFERENCES

Milne, W. E. (1949). *Numerical Calculus*, pp. 15–29. Princeton University Press, Princeton, New Jersey.

Morse, P. M. and Feshbach, H. (1953). *Methods of Theoretical Physics*, p. 1466. McGraw-Hill, New York.

Shapiro, I. I., Ash, M. E., Ingalls, R. P., Smith, W. B., Campbell, D. B., Dyce, R. B., Jurgens, R. F., and Pettengill, G. H. (1971). *Physical Review Letters*, **26**, 1132.